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A DIAGRAM FOR THE SOLUTION OF TRIANGLES.

BY C. O. TUCKEY.

IN February 1939 the *Gazette* published a " diagram for the study and solution of triangles ", in which there was a one-to-one correspondence between each point within a certain area and a triangle of specified shape. The diagram which it is the purpose of this article to explain is even simpler in theory and is perhaps more effective for the solution of triangles, though it has not the interesting features of the one-to-one correspondence between a point and a triangle.

To construct this diagram a square 10 inches each way is taken and the sides are each marked with *two* scales :

(i) taking 10 inches as the unit, each side is marked uniformly by decimals from 0 to 1, the lengths representing in use both the lengths of the sides of triangles and the *sines* of the angles ;

(ii) each side is also marked, not uniformly, with angles from 0° to 90°, the angles being placed according to their sines, so that, for example, the line which was marked ·5 is also marked 30°. A glance at Fig. 1 will show the arrangement.

The diagram should be drawn on squared paper so that the rulings of the paper show two-place decimals. It might be convenient to rule the angle-scale across this in red ink.

Taking the lengths as coordinates the diagram is completed by drawing the curves $\arcsin x + \arcsin y = \theta$, where θ is 10°, 20°, 30°, 40°, etc. These curves are easily drawn (far more easily than those of the other triangle diagram) by drawing through the corners formed by the intersections of the lines of the angle scale. Those shown in Fig. 1 are for $\theta = 40°$, 50° and 140°. Of these, the first is drawn through the points (0°, 40°), (10°, 30°), (20°, 20°), (30°, 10°), (40°, 0°).

The figure shows how to solve a triangle in which sides 82, 48 contain an angle of 130°. The point A is taken whose coordinates are (·82, ·48) and this is joined to the origin by a line which cuts the curve $\arcsin x + \arcsin y = 50°$ at the point B. If θ, ϕ are the angles to be found, the point B provides the solution, for if the coordinates of B on the *angle* scale are θ, ϕ, then both $\sin\theta/\sin\phi = 82/48$ and $\theta + \phi = 50°$. The solution can be read directly on the

angle scale or sin θ, sin ϕ can be read on the uniform scale provided by the squared paper. Here very roughly $\sin\theta = \cdot 53$ and $\sin\phi = \cdot 31$.

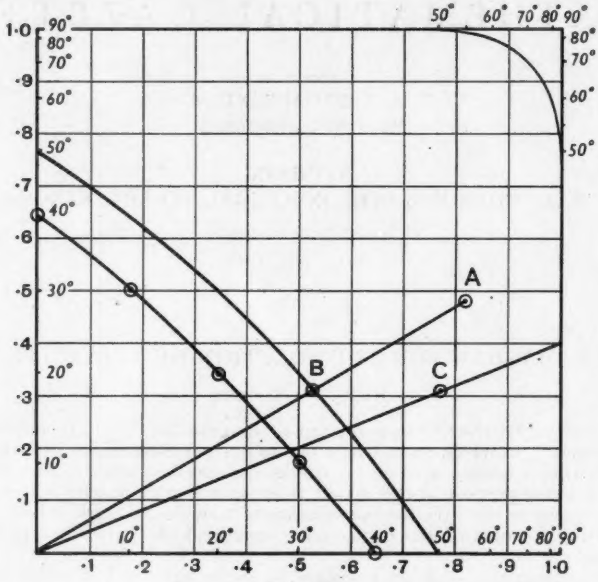


FIG. 1.

Fig. 2 shows a small portion of a diagram drawn on the suggested scale of 10 inches to the unit with the curves drawn for every degree. The solution shown is for a triangle in which sides 76·5 and 20 contain an angle of 125°. The line AB if produced would pass through the origin and B gives the other angles, giving their sines as ·698 and ·184, and so the angles themselves as 44° 18′ and 10° 36′ (the correct values being 44° 27′ and 10° 33′; had the first angle been cooked to give the known sum of 55°, the answer would have been much improved).

The solution is easily completed from the diagram. If in Fig. 1 the point C is taken having the y of B and $x = \sin 50°$ and if C is joined to the origin, then for this line $y/x = \sin\phi/\sin(\theta+\phi)$. Hence the third side is read off as the x of the point when the y is that of A. This point is (118, 48). If, as is the case here, the point is outside the square it is easy to double the value of x corresponding to $y = 24$, which is 59.

The diagram is equally adapted to the A.A.S. and A.S.S. cases of solution. For the A.A.S. case it is merely a matter of starting with the point B and reading the answer at the point A. It is less suitable for the S.S.S. case (though it can be made to fit it by a bit of juggling), but this hardly matters in practice as angles are always measured out of doors in preference to lines. Here this diagram has the advantage over the other one mentioned above as that one was at its worst for the S.A.S. case, if the unknown side was the largest one.

In Fig. 2 the curves for each degree are adequately spaced. This, however, would only be the case for part of the diagram, and they would be much closer together for smaller contained angles. If, however, 20 inches was taken as the unit, the curves for each degree would be satisfactorily spaced for contained angles down to 25°, or perhaps lower. For small contained angles the top right-hand corner of the diagram would need to be enlarged. It is hard

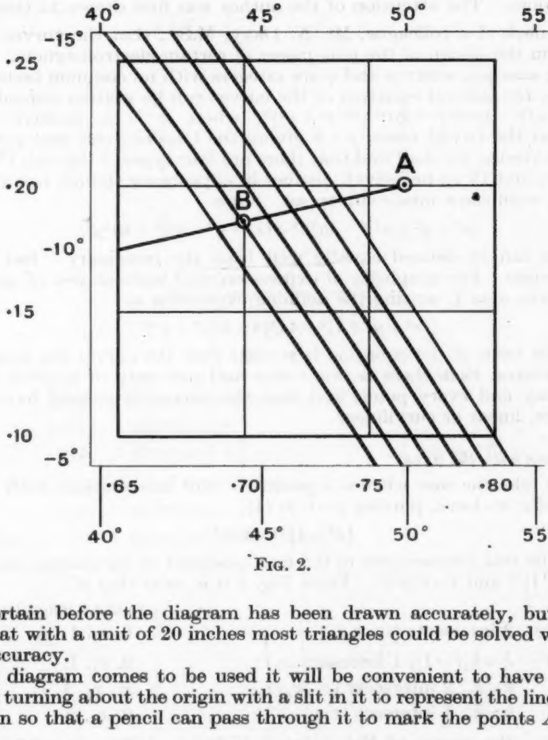


FIG. 2.

to be certain before the diagram has been drawn accurately, but it seems likely that with a unit of 20 inches most triangles could be solved with three-figure accuracy.

If the diagram comes to be used it will be convenient to have a strip of celluloid turning about the origin with a slit in it to represent the lines through the origin so that a pencil can pass through it to mark the points A and B.

C. O. T.

GLEANINGS FAR AND NEAR.

1419. It is, however, perhaps curious that the committee should have adopted the proposition that when certificates are equal to the same requirements they are equal to one another!—*Manchester Guardian*, Jan. 3rd, 1939, in an interview given by Dr. J. E. Myers on the *Report of the Consultative Committee on Secondary Education*. [Per Mr. P. J. Harris.]

ON SOME GENERAL OVALS OF CASSINIAN TYPE.

By H. Gwynedd Green.

The following discussion of the curves $r_1 r_2 = cr^n$, where r_1, r_2, r are the distances of a point from the " foci " $\pm a$, 0 and the origin, c a positive constant and n rational (integral or fractional, positive or negative), provides a good example of extensive knowledge of general curves obtained by elementary considerations. The attention of the author was first drawn to these curves by the remark of a colleague, Mr. N. Davy, M.Sc., that the curves $r_1 r_2 = cr^{\frac{1}{3}}$ appeared in the design of the pole-pieces of certain electromagnets.

Writing n as p/q where p and q are integers with no common factors and q is positive, the general equation of the curves can be written immediately as $\{(x^2+y^2+a^2)^2 - 4a^2x^2\}^q = ka^{4q-2p}\{x^2+y^2\}^p$, where k is a positive number. Apart from the trivial cases, $p=0$ giving the Cassini ovals and $q=0$ giving concentric circles, we shall find that there are four types, I ($2q>p$), II ($2q=p$), III ($2q<p$) and IV (p negative), distinct in appearance though not altogether unrelated, with some minor sub-types. Since

$$(x^2+y^2+a^2)^2 - 4a^2x^2 = (x^2+y^2-a^2)^2 + 4a^2y^2,$$

the curves can be defined equally well from the imaginary " foci " 0, $\pm ai$ and the origin. For simplicity of expression, and without loss of generality, we shall take a as 1, writing the defining expression as

$$\{(x^2+y^2+1)^2 - 4x^2\}^q = k\{x^2+y^2\}^p. \qquad \text{*}(1)$$

From the form of the equation it is clear that the curves are symmetrical about the axes, that there is one curve and one only of a given n-family through any and every point, and that the curves in general have no real asymptotes, linear or curvilinear.

Intersections with the axes.

We first take the case when n is positive. For intersections with the positive axis of x we have, putting $y=0$ in (1),

$$\{x^2-1\}^{2q} = kx^{2p} \qquad (2)$$

Consider the real intersections in the first quadrant of the graphs, in x^2 and z, of $z=(x^2-1)^{2q}$ and $z=k(x^2)^p$. From Fig. 1 it is clear that if

		relative magnitudes:
$2q>p$:	2 intersects (x_1, x_2) ;	$0, \underline{x_1}, 1, \underline{x_2}$.
$2q=p$:	$k=k_1(=1)$, 1 intersect (x_1) ;	$0, \underline{x_1}, 1$.
	$k<k_1$, 2 intersects (x_2, x_3) ;	$x_1, \underline{x_2}, 1, \underline{x_3}$.
	$k>k_1$, 1 intersect (x_4).	$0, \underline{x_4}, x_1$.

$2q<p$: the curves of Fig. 1 touch if $2q(x^2-1)^{2q-1} = kp(x^2)^{p-1}$ satisfies equation (2), *i.e.* at $x=X$ where

$$(X^2-1)/2q = X^2/p \text{ or } X^2 = p/(p-2q)$$

and
$$k=k_2=\left(\frac{2q}{p}\right)^{2q}\left(\frac{p-2q}{p}\right)^{p-2q}.$$

	relative magnitudes:
$k=k_2$, 1 and 2 coincident (x_1, X) ;	$0, \underline{x_1}, 1, \underline{X}$.
$k<k_2$, 3 intersects (x_2, x_3, x_4) ;	$x_1, \underline{x_2}, 1, \underline{x_3}, X, \underline{x_4}$.
$k>k_2$, 1 intersect (x_5) ;	$0, \underline{x_5}, x_1$.

* For numerical calculation a convenient form is $4x^2 = (r^2+1)^2 - k^{\frac{1}{q}}(r^2)^{\frac{p}{q}}$.

For intersections with the positive axis of y we have, similarly,

$$(y^2+1)^{2q}=k(y^2)^p,$$

and if

$2q>p$: the curves of Fig. 2 touch if $2q(y^2+1)^{2q-1}=kp(y^2)^{p-1}$ satisfies the equation in k, y, *i.e.* at $y=Y$ where

$$(Y^2+1)/2q=Y^2/p \text{ or } Y^2=p/(2q-p)$$

and $$k=k_3=\left(\frac{2q}{p}\right)^{2q}\left(\frac{p}{2q-p}\right)^{2q-p}.$$

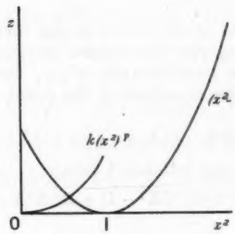


FIG. 1.

		relative magnitudes:
	$k=k_3$, 2 coincident intersects (Y);	
	$k<k_3$, 0 intersects;	
	$k>k_3$, 2 intersects (y_1, y_2);	0, $\underline{y_1}$, Y, $\underline{y_2}$.
$2q=p$:	$k\leqslant k_1$ $(=1)$, 0 intersects;	
	$k>k_1$, 1 intersect;	
$2q<p$:	1 intersect in all cases.	

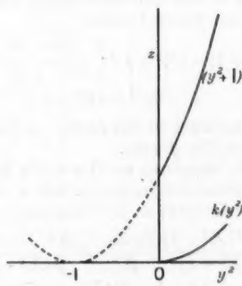


FIG. 2.

Taking now the case with n negative, we replace the curves through the origin in Figs. 1, 2 by "hyperbolic" curves $z(x^2)^{-p}=k$, $z(y^2)^{-p}=k$. From similar treatment as the preceding, we obtain for

$k=k_4=\left(\frac{2q}{-p}\right)^{2q}\left(\frac{-p}{2q-p}\right)^{2q-p}$,	2 coincident intersections at $X_1{}^2=-p/(2q-p)$ and another (x_1);	relative magnitudes: $0, \underline{X_1}, 1, \underline{x_1}$.
for $k>k_4$,	1 intersect (x_2);	$x_1, \underline{x_2}$.
for $k<k_4$,	3 intersects (x_3, x_4, x_5).	$0, \underline{x_3}, X, \underline{x_4}, 1, \underline{x_5}, x_1$.

and for intersects on the y-axis, 1 in all cases.

We shall term the appropriate curves corresponding to the values k_1, k_2, k_3, k_4, " the critical curves ", and these values of k the " critical values ".

The form of a critical curve at its point of double intersection with an axis.

We shall take the equation of the curve in the form $r_1{}^2r_2{}^2=k^{\frac{1}{q}}(r^2)^{\frac{p}{q}}$ * where k has a critical value giving rise to a point of double intersection $(X, 0)$. Transferring the origin to new parallel axes (x', y') through $(X, 0)$ and writing R^2 for $x'^2+y'^2$, we have for the equation of the curve

$$[\{X^2+1+2x'X+R^2\}-4\{X+x'\}^2]=k^{\frac{1}{q}}[X^2+2x'X+R^2]^{\frac{p}{q}}.$$

On expanding we obtain for the left-hand side,

$$(X^2-1)^2+4x'X(X^2-1)+4x'^2(X^2-1)+4x'R^2X+2R^2(X^2+1)+R^4,$$

and for the right-hand side,

$$k^{\frac{1}{q}}(X^2)^{\frac{p}{q}}\left[1+\frac{2x'}{X}+\frac{R^2}{X^2}\right]^{\frac{p}{q}}$$

$$=k^{\frac{1}{q}}(X^2)^{\frac{p}{q}}\left[1+\frac{p}{q}\left\{\frac{2x'}{X}+\frac{R^2}{X^2}\right\}+\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{2}\left\{\frac{4x'^2}{X^2}+\frac{4x'R^2}{X^3}+\frac{R^4}{X^4}\right\}+\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{6}\left\{\frac{8x'^3}{X^3}+\dots\right\}+\dots\right].$$

Using the relations $k=(X^2-1)^{2q}/(X^2)^p$, $p/q=2X^2/(X^2-1)$, and comparing terms (transferring all terms to the left-hand side), those in 1 and x' disappear automatically and the squared terms become

$$4x'^2(X^2-1)+2R^2(X^2+1)-(X^2-1)^2\left[\frac{2X^2}{X^2-1}\frac{R^2}{X^2}+\frac{X^2(X^2+1)}{(X^2-1)^2}\frac{4x'^2}{X^2}\right]$$
$$=-8x'^2+4R^2=4(y'^2-x'^2),$$

or the tangents are equally inclined to the axes. A like result will be obtained for the tangents at a node on the y-axis.

Changing the axes to the tangents at the node by writing $x_1+y_1=\sqrt{2}x'$, $x_1-y_1=\sqrt{2}y'$, so that the original origin lies in the $-\ -$ quadrant, the squared terms become $-8x_1y_1$, and the term in $x_1{}^3$ becomes

$$2\sqrt{2}x_1{}^3X-(X^2-1)^2\left[\frac{X^2(X^2+1)}{(X^2-1)^2}\frac{2\sqrt{2}x_1{}^3}{X^3}+\frac{X^2(X^2+1)2}{3(X^2-1)^3}\frac{2\sqrt{2}x_1{}^3}{X^3}\right]$$
$$=2\sqrt{2}x_1{}^3\left[X-\frac{X^2+1}{X}-\frac{2(X^2+1)}{3X(X^2-1)}\right]=2\sqrt{2}\,\frac{1-5X^2}{3X(X^2-1)}\,x_1{}^3,$$

* No ambiguity arises from the root, as we must take the real value satisfying the value $(X, 0)$. The subsequent work is valid for all values of p/q, including negative ones.

and the first approximation of the equation of the branch touching $y_1=0$ is therefore

$$2\sqrt{2}y_1+\frac{1-5X^2}{3X(1-X^2)}x_1^2=0. \qquad (3)$$

The curvature at the node is therefore towards the origin except when X^2 lies between 1 and 1/5, that is, when p is negative and $-2p>q$.

The loci of the points of contact of tangents parallel to the axes.

From equation (1) $\dfrac{dy}{dx}=0$ when

$$4q\{(x^2+y^2+1)^2-4x^2\}^{q-1}(x^2+y^2-1)x=2pk\{x^2+y^2\}^{p-1}x,$$

i.e. at the intersections of the curve with $x=0$ (except at double points, when the analysis fails) and

$$p\{(x^2+y^2+1)^2-4x^2\}=2q(x^2+y^2-1)(x^2+y^2). \qquad (4)$$

Putting $x^2+y^2=r^2$ in this last equation and simplifying,

$$(2q-p)r^4-2(q+p)r^2+4px^2-p=0,$$

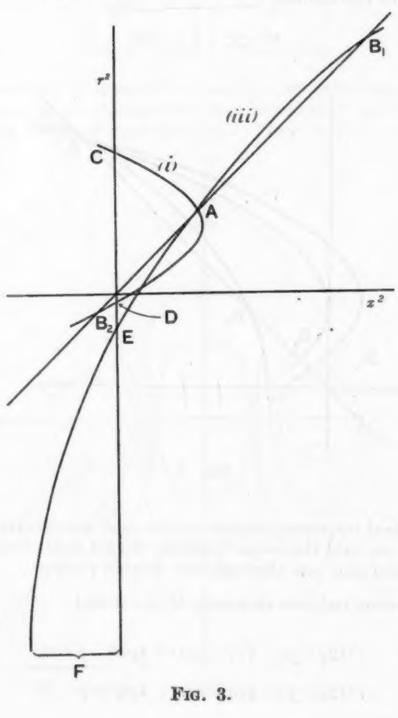


Fig. 3.

or $$\left[r^2 - \frac{q+p}{2q-p}\right]^2 = -\frac{4p}{2q-p}x^2 + \frac{q^2+4qp}{(2q-p)^2}, \qquad \text{.....................(5)}$$

except in the case $2q = p$ (Type II) when this curve reduces to the hyperbola $x^2 - 3y^2 = 1$ (and strictly $1/(x^2+y^2) = 0$). The behaviour of the loci (4) can be conveniently demonstrated by those parts of the parabolic graph of (5) obtained by plotting r^2 against x^2 which lie in the positive octant between $r^2 = x^2$ and $x^2 = 0$. From (4) the graphs cut $r^2 = x^2$ (*i.e.* $y^2 = 0$) at the points A and B where $x^2 = 1$ and $-\dfrac{p}{2q-p}$, *i.e.* the locus passes through the foci and double points of the critical curve on the x-axis : Fig. 3 shows typical cases for (i) $2q > p$ (Type I) and (iii) $2q < p$ (Type III)—Type II has already been disposed of. From (i) it will be seen that the required locus consists of a closed oval meeting the x-axis in the foci only (point A in Fig. 3) and meeting the y-axis in real points (point C) and from (iii) that the locus consists of ovals (two) not meeting the y-axis, but each passing through a focus (A) and a real double point of the critical curve (B_1). Fig. 4 shows typical cases for negative p, and it will be seen that Type IV has three sub-types: in cases (a) $q < 4(-p)$, and this "tangent" locus consists of ovals through the foci and double points (A, B_a) and not meeting the y-axis ; in case (b) $q = 4(-p)$, and this locus reduces to the circles

$$x^2 + y^2 - \tfrac{1}{3} = \pm\tfrac{2}{3}x$$

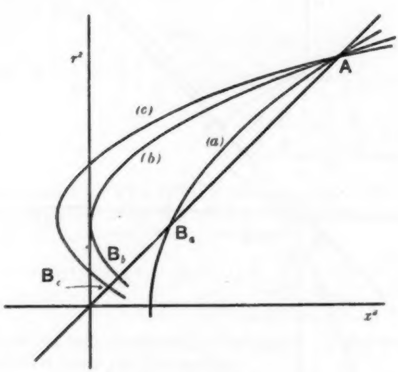


FIG. 4.

through the foci and opposite double points and intersecting on the y-axis ; in case (c) $q > 4(-p)$, and the locus consists of two ovals enclosing the origin, one through the foci and one through the double points.

For $\dfrac{dx}{dy} = 0$ the locus reduces similarly to $y = 0$ and

$$r^4(2q-p) + 2(q-p)r^2 + 4px^2 - p = 0,$$

i.e. $$r^4(2q-p) + 2(q+p)r^2 - 4py^2 - p = 0,$$

or $$\left(r^2+\frac{q+p}{2q-p}\right)^2=\frac{4p}{2q-p}\,y^2+\frac{q^2+4qp}{(2q-p)^2}, \qquad \text{.......................(6)}$$

except in the case $2q=p$ (Type II) when this curve reduces to the hyperbola $3x^2-y^2=1$ (and, strictly, $1/(x^2+y^2)=0$).

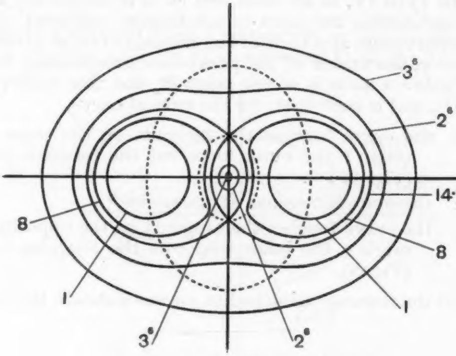


FIG. 5. Type I ($p=1$, $q=3$).

These equations can be treated in exactly the same way as equations (5), and if, in Figs. 3 and 4, we changed the marked axes x^2 and r^2 to $-y^2$ and $-r^2$, the graphs become those of equations (6). They cut $r^2=y^2$ (*i.e.* $x^2=0$) at the

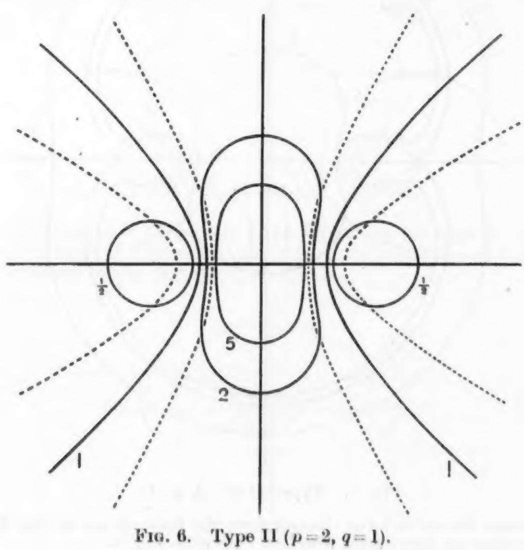


FIG. 6. Type II ($p=2$, $q=1$).

points $y^2 = -1$ and $p/(2q-p)$—*i.e.* the locus passes through the imaginary foci (A) and the double points of the critical curve on the y-axis (B). In Type I the locus consists of an oval through the real double points B_2, and cutting the x-axis in the real points D, in Type III we have ovals on either side of the y-axis and not cutting it, but cutting the x-axis in pairs of real points (E, F). In Type IV, in all cases, the locus is completely imaginary.

Fig. 4 also demonstrates the point of subdivision indicated by the change in the sign of the curvature at a node of the critical curve, as given in equation (3). When $-2p = q$, the vertex of the parabola corresponding to the curved portion of the $dy/dx = 0$ locus is at the point B_a and this subdivision in type occurs in Type IV, and is such that, for the critical curve,

if $q > -2p$: the curve touches the tangents on the same sides as the origin : the outer loops cut the tangents in real points (Fig. 9, 10).

$q = -2p$: there are inflections at the nodes.*

$q < -2p$: the curve touches the tangents on the opposite sides to the origin : the inner loop cuts the tangents in real points (Fig. 8).

In Figs. 5 to 10 the numbers attached to curves indicate the corresponding values of k.

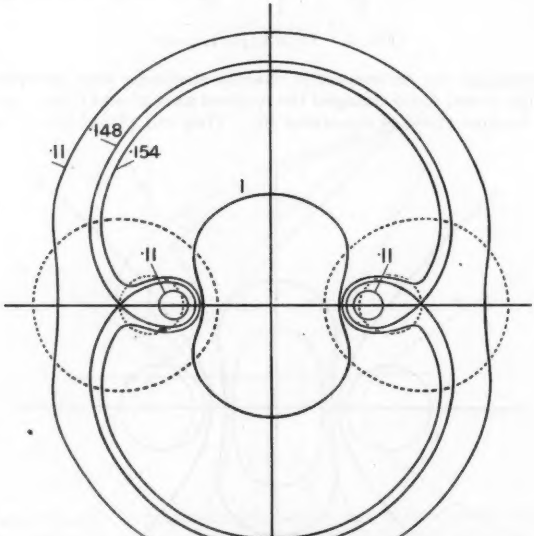


Fig. 7. Type III ($p = 3$, $q = 1$).

* At this stage the dy/dx locus changes from the form shown in Fig. 8 to " bean-shaped ", to break up into circles when $q = -4p$ as in Fig. 9.

Inversion.

If the curve $(r_1r_2)^q = k^{\frac{1}{2}}r^p$ is inverted with respect to the circle $x^2+y^2=1$, the equation of the new curve is

$$\left(\frac{r_1r_2}{r^2}\right)^q = k^{\frac{1}{2}}\frac{1}{r^p}, \text{ or } (r_1r_2)^q = k^{\frac{1}{2}}r^{2q-p}.$$

The inverse of a curve of Type I is therefore of Type I ($2q>p$, $2q>2q-p$): the inverse of a curve of Type II is a Cassini Oval ($q=1$, $p=2$, $2q-p=0$):

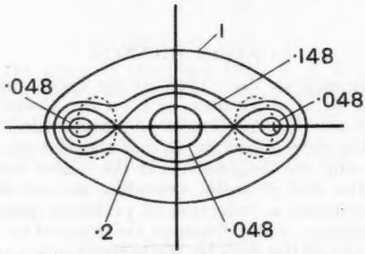


FIG. 8. Type IV*a* ($p=-1$, $q=1$).

the inverse of Type III is of Type IV ($2q<p$, $2q-p$ negative), and of Type IV is of Type III ($2q<2q-p$), and the point of division $q=-2p$ in Type IV has

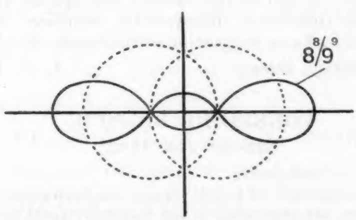


FIG. 9. Type IV*b* ($p=-1$, $q=4$).

a corresponding point of division in Type III concerned with the intersections other than the node of the critical curve with the circle $x^2+y^2=X(x+y)$ (the circle of curvature in the case $q=2$, $p=5$).

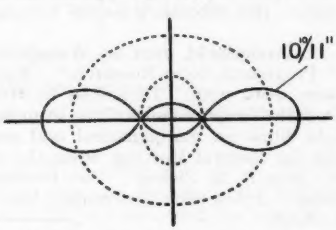


FIG. 10. Type IV*c* ($p=-1$, $q=5$).

From the r, x forms of the equations of the curved portions of $dy/dx=0$, $dx/dy=0$, it is evident at once that these portions do not intersect in real finite points, and hence the only real finite double points are those already established on the critical curves I, III, IV on the axes.

Figs. 5, 6, 7, 8 give examples of families of curves of Types I, II, III, IVa : the broken lines are the curved portions of $dy/dx=0$, $dx/dy=0$. Figs. 9, 10 give the critical curves for IVb, IVc, and the curved portions of their $dy/dx=0$ loci. It will be seen that the differences between the sub-types of Type IV (apart from the change at $q=2$; $p=-1$ in (a)) are mainly those of relative size of the three areas of the critical curve and of the direction of retreat from the node. H. G. G.

LONDON BRANCH.

A MEETING was held on 30th May, 1942, at the Polytechnic, Regent Street. A paper was read by Mr. A. J. G. May, of Isleworth County School, on " The Analysis of Data for the Construction of Quadrilaterals ".

The quadrilateral is determined by five of the elements consisting of the lengths of the sides and the diagonals, and the angles between any pairs of these. There are five sets of cases, according as one distance or more is given. Each set comprises a collection of problems which can be grouped into classes. For example, if the distances are denoted by a, b, c, α, β, γ, and the angles between two of the lines by the corresponding couple, then for the set " Three sides and two angles " there are 2100 cases divided into 25 groups of which one is a, b, c, $a\alpha$, $a\beta$. Constructions were given for a dozen of the more interesting cases.

The next meeting will be held at the Polytechnic, Regent Street, on 12th March, 1943, at 3 p.m. It will be the Annual Meeting at which arrangements for the future will be discussed, followed by members' topics. The Hon. Secretary will be glad to have suggestions of subjects for discussion.

30 Manor Gardens, Purley, Surrey. A. J. TAYLOR, *Hon. Sec.*

YORKSHIRE BRANCH.

REPORT FOR 1942.

Three meetings have been held.

7th March. Mr. Montagnon of Leeds Grammar School gave a paper entitled " What changes, if any, are desirable in the mathematical syllabus and outlook for the Higher School Certificate examination, particularly in applied mathematics ". These should be in four groups : (i) Scholarship papers in pure or applied mathematics for Mathematics Honours students ; (ii) pass and scholarship papers in pure or applied mathematics for Science Honours students ; (iii) pass and scholarship papers in pure and applied mathematics for Honours students in other subjects ; (iv) subsidiary papers for pass students in other subjects.

6th June. Dr. G. H. Archerhold, now at Wakefield Grammar School, gave a paper entitled " Progress in Solar Research ". Sun-spots, solar energy and nuclear physics were dealt with. Then Miss E. Hudson opened a discussion on " The mathematical requirements of the secondary school entrant".

10th October. Eight films on mathematical and scientific topics were shown. These followed the General Meeting, when the officers were elected for 1942–3 : *President* : Miss F. M. Pickup ; *Vice-Presidents* : Mr. W. G. L. Sutton, Mr. H. H. Watts ; *Acting Hon. Secretary* : Miss E. Hudson ; *Hon. Treasurer* : Miss N. K. Rose.

E. HUDSON, *Hon. Sec.*

THEOREMS ON QUADRIPLANAR COORDINATES.

By R. T. Robinson.

1. *The angle between two lines.*

Given a tetrahedron of reference $ABCD$, let λ, μ, ν, π be the cosines of the angles made by a line with the perpendiculars to the faces BCD, ACD, ABD, ABC respectively, these perpendiculars being drawn all inwards or all outwards ; where

$$A\lambda_1 + B\mu_1 + C\nu_1 + D\pi_1 = 0,$$
$$A\lambda_2 + B\mu_2 + C\nu_2 + D\pi_2 = 0,$$

A, B, C, D being the areas of the faces BCD, ... If P_1 $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ and P_2 $(\alpha_2, \beta_2, \gamma_2, \delta_2)$ are any two points, the lines PP_1 and PP_2 can be written

$$(\alpha - \alpha_1)/\lambda_1 = (\beta - \beta_1)/\mu_1 = (\gamma - \gamma_1)/\nu_1 = (\delta - \delta_1)/\pi_1 = \theta_1,$$
$$(\alpha - \alpha_2)/\lambda_2 = (\beta - \beta_2)/\mu_2 = (\gamma - \gamma_2)/\nu_2 = (\delta - \delta_2)/\pi_2 = \theta_2 ;$$

to find the cosine of the angle between PP_1 and PP_2.

It can be shown that

$$-9V^2 . P_1P_2^2 = \Sigma c^2AB(\alpha_1 - \alpha_2)(\beta_1 - \beta_2),$$

and hence $\quad -9V^2 . PP_1^2 = \Sigma c^2AB(\alpha - \alpha_1)(\beta - \beta_1)$

$$= \theta_1^2 . \Sigma c^2 AB\lambda_1\mu_1 ;$$

similarly $\quad -9V^2 . PP_2^2 = \theta_2^2 . \Sigma c^2AB\lambda_2\mu_2 ;$

also $\quad \alpha_1 - \alpha_2 = -(\theta_1\lambda_1 - \theta_2\lambda_2),\ \beta_1 - \beta_2 = -(\theta_1\mu_1 - \theta_2\mu_2)$, etc.

Thus $\quad -9V^2 . P_1P_2^2 = \Sigma c^2 . AB(\theta_1\lambda_1 - \theta_2\lambda_2)(\theta_1\mu_1 - \theta_2\mu_2)$

$$= -9V^2 . PP_1^2 - 9V^2 . PP_2^2 - \theta_1\theta_2\Sigma c^2 . AB(\lambda_1\mu_2 + \lambda_2\mu_1).$$

Hence $\quad 9V^2(PP_1^2 + PP_2^2 - P_1P_2^2)$

$$= \{-9V^2 . PP_1 . PP_2\Sigma c^2AB . (\lambda_1\mu_2 + \lambda_2\mu_1)\}/\surd(-\Sigma c^2AB\lambda_1\mu_1)\surd(-\Sigma c^2AB\lambda_2\mu_2).$$

Thus the cosine of the angle between the lines

$$= (PP_1^2 + PP_2^2 - P_1P_2^2)/2PP_1 . PP_2$$
$$= -\{\Sigma c^2AB(\lambda_1\mu_2 + \lambda_2\mu_1)\}/2\surd(-\Sigma c^2AB\lambda_1\mu_1) . \surd(-\Sigma c^2AB\lambda_2\mu_2).$$

In two dimensions the cosine of the angle between the straight lines

$$(\alpha - \alpha_1)/\lambda_1 = (\beta - \beta_1)/\mu_1 = (\gamma - \gamma_1)/\nu_1,$$
$$(\alpha - \alpha_2)/\lambda_2 = (\beta - \beta_2)/\mu_2 = (\gamma - \gamma_2)/\nu_2$$

can be shown to be

$$\{\Sigma c(\lambda_1\mu_2 + \lambda_2\mu_1)\}/2\surd(-\Sigma c\lambda_1\mu_1) . \surd(-\Sigma c\lambda_2\mu_2).$$

2. *A plane perpendicular to a given line.*

To find the equation of the plane drawn through $P_1(\alpha_1, \beta_1, \gamma_1, \delta_1)$ perpendicular to the line

$$(\alpha - \alpha_0)/\lambda = (\beta - \beta_0)/\mu = (\gamma - \gamma_0)/\nu = (\delta - \delta_0)/\pi. \quad \text{......................(i)}$$

If $P(\alpha, \beta, \gamma, \delta)$ is any point on the plane, the cosines (λ, μ, ν, π) of the line PP_1 are proportional to $(\alpha - \alpha_1)$, etc., and this line PP_1 must be perpendicular to (i). Hence

$$\Sigma c^2AB\{\lambda(\beta - \beta_1) + \mu(\alpha - \alpha_1)\} = 0 ;$$

that is,

$$(\alpha-\alpha_1)(c^2AB\mu+b^2AC\gamma+d^2AD\pi)+(\beta-\beta_1)(c^2AB\lambda+d^2BC\gamma+e^2BD\pi)$$
$$+(\gamma-\gamma_1)(b^2AC\lambda+a^2BC\mu+f^2CD\pi)+(\delta-\delta_1)(d^2AD\lambda+e^2BD\mu+f^2CD\nu)=0.$$

Thus *planes perpendicular to* (i) *are parallel to the plane*

$$A\alpha(c^2B\mu+b^2C\gamma+d^2D\pi)+B\beta(c^2A\lambda+a^2C\gamma+e^2D\pi)$$
$$+C\gamma(b^2A\lambda+a^2B\mu+f^2D\pi)+D\delta(d^2A\lambda+e^2B\mu+f^2C\gamma)=0.$$

This plane passes through the centre $(V_1/A,\ V_2/B,\ V_3/C,\ V_4/D)$ of the sphere $ABCD$, for on substituting these coordinates in the left-hand side of the preceding equation we have

$$A\lambda(c^2V_2+b^2V_3+d^2V_4)+B\mu(c^2V_1+a^2V_3+e^2V_4)$$
$$+C\nu(b^2V_1+a^2V_2+f^2V_4)+D\pi(d^2V_1+e^2V_2+f^2V_3)$$
$$=(A\lambda+B\mu+C\gamma+D\pi)2V\rho^2$$
$$=0, \text{ since } A\lambda+B\mu+C\gamma+D\pi=0.$$

In two dimensions it can be shown that the equation of the straight line through $(\alpha_1, \beta_1, \gamma_1)$ perpendicular to the straight line whose equation is

$$(\alpha-\alpha_0)/\lambda=(\beta-\beta_0)/\mu=(\gamma-\gamma_0)/\nu$$

is
$$(\alpha-\alpha_1)(c\mu+b\nu)+(\beta-\beta_1)(c\lambda+a\nu)+(\gamma-\gamma_1)(a\mu+b\lambda)=0.$$

This is parallel to the line whose equation is

$$\alpha(c\mu+\beta\nu)+\beta(c\lambda+a\nu)+\gamma(a\mu+b\lambda)=0,$$

and this line passes through the centre of the circle ABC, since $a\lambda+b\mu+c\nu=0$.

3. *A line perpendicular to a given plane.*

To show that if the straight line

$$(\alpha-\alpha_0)/\lambda=(\beta-\beta_0)/\mu=(\gamma-\gamma_0)/\nu=(\delta-\delta_0)/\pi$$

is perpendicular to the given plane

$$l\alpha+m\beta+n\gamma+p\delta=0,$$

then
$$\lambda:\mu:\nu:\pi$$
$$=(-l+m\cos AB+n\cos AC+p\cos AD):$$
$$(l\cos AB-m+n\cos BC+p\cos BD):$$
$$(l\cos AC+m\cos BC-n+p\cos CD):$$
$$(l\cos AD+m\cos BD+n\cos CD-p).$$

The equation of any plane through AB is $n_1\gamma+p_1\delta=0$. If this is perpendicular to the given plane, then

$$nn_1+pp_1-mn_1\cos BC-mp_1\cos BD-(np_1+n_1p)\cos CD$$
$$-lp_1\cos AD-ln_1\cos AC=0$$

or
$$n_1(l\cos AC+m\cos BC+p\cos CD-n)$$
$$+p_1(l\cos AD+m\cos BD+n\cos CD-p)=0.$$

Thus the equation to the plane is

$$\gamma/(l\cos AC+m\cos BC+p\cos CD-n)$$
$$=\delta/(l\cos AD+m\cos BD+n\cos CD-p).$$

Similarly the equation to the plane through AC perpendicular to the given plane is

$$\beta/(l\cos AB - m + n\cos BC + p\cos BD)$$
$$= \delta/(l\cos AD + m\cos BD + n\cos CD - p).$$

Thus the equations to the straight line through A perpendicular to the given plane are

$$\begin{aligned}\beta/(l\cos AB - m + n\cos BC + p\cos BD) &\\ = \gamma/(l\cos AC + m\cos BC + p\cos CD - n) &\\ = \delta/(l\cos AD + m\cos BD + n\cos CD - p) &\\ = (\alpha - 3V/A)/x, &\end{aligned}$$

where $Ax + B(l\cos AB - m + n\cos BC + p\cos BD)$
$$+ C(l\cos AC + m\cos BC + p\cos CD - n)$$
$$+ D(l\cos AD + m\cos BD + n\cos CD - p) = 0,$$

that is, $Ax + l(B\cos AB + C\cos AD + D\cos AD)$
$$+ m(-B + C\cos BC + D\cos BD)$$
$$+ n(B\cos BC - C + D\cos CD)$$
$$+ p(B\cos BD + C\cos CD - D) = 0,$$

or $$Ax + Al - mA\cos AB - nA\cos AC - pA\cos AD = 0,$$

that is, $$x = -l + m\cos AB + n\cos AC + p\cos AD.$$

We have thus found a line with the required direction; to complete the proof it only remains to draw a parallel line through $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, giving the result stated at the beginning of this section.

4. *A plane perpendicular to the straight line*

$$(\alpha - \alpha_1)/\lambda = (\beta - \beta_1)/\mu = (\gamma - \gamma_1)/\nu = (\delta - \delta_1)/\pi$$

must be parallel to the plane whose equation is

$$\alpha(c^2AB\mu + b^2AC\nu + d^2AD\pi) + \beta(c^2AB\lambda + a^2BC\nu + e^2BD\pi) + \ldots = 0. \quad \text{......(i)}$$

Thus when $$\lambda = -l + m\cos AB + n\cos AC + p\cos AD,$$
$$\mu = l\cos AB - m + n\cos BC + p\cos BD,$$
$$\vdots$$

the plane (i) should be parallel to the plane whose equation is

$$l\alpha + m\beta + n\gamma + p\delta = 0.$$

To prove that this is so, we take the coefficient of α in (i) with these values of λ, μ, ... and, arranging in terms of l, m, n, p, we obtain

$$\begin{aligned}l(c^2AB\cos AB + b^2AC\cos AC + d^2AD\cos AD) &\\ - m(c^2AB - b^2AC\cos BC - d^2AD\cos BD) &\\ - n(b^2AC - c^2AB\cos BC - d^2AD\cos CD) &\\ - p(d^2AD - c^2AB\cos BD - b^2AC\cos CD). &\end{aligned}$$

In this the coefficient of l is $18(V^2 - VV_1)$;

the coefficient of m is

$$-(A/B)(c^2B^2 - b^2BC\cos BC - d^2BD\cos BD)$$
$$= -(A/B)\,.\,18VV_2,$$

the coefficients of n and p being similarly

$$-(A/C)\,.\,18VV_3, \text{ and } -(A/D)\,.\,18VV_4.$$

Thus $\quad c^2AB\mu + b^2AC\nu + d^2AD\pi = 18V^2l - 18VA(V_1/A + V_2/B + V_3/C + V_4/D).$

Similarly

$$c^2AB\lambda + a^2BC\nu + e^2BD\pi = 18V^2m - 18VB(V_1/A + V_2/B + V_3/C + V_4/D),$$
$$b^2AC\lambda + a^2BC\mu + f^2CD\pi = 18V^2n - 18VC(V_1/A + V_2/B + V_3/C + V_4/D),$$
$$d^2AC\lambda + e^2BD\mu + f^2CD\nu = 18V^2p - 18VD(V_1/A + V_2/B + V_3/C + V_4/D).$$

Thus equation (i) becomes

$$18V^2(l\alpha + m\beta + n\gamma + p\delta) - 18V\,.\,\Sigma(V_1/A)\,.\,(A\alpha + B\beta + C\gamma + D\delta) = 0,$$

that is, it is the equation of a plane parallel to the plane

$$l\alpha + m\beta + n\gamma + p\delta = 0.$$

5. *Three perpendicular straight lines.*

In what follows, we suppose that l, m, n, p are the cosines of the angles made with the perpendiculars to the faces BCD, ACD, ABD, ABC of the tetrahedron, and that in all cases

$$Al + Bm + Cn + Dp = 0.$$

Let three perpendicular straight lines through $O(\alpha_0, \beta_0, \gamma_0, \delta_0)$ be taken as Ox, Oy, and Oz. Their equations are

$$(\alpha - \alpha_0)/l_r = (\beta - \beta_0)/m_r = (\gamma - \gamma_0)/n_r = (\delta - \delta_0)/p_r\,; \quad (r = 1, 2, 3)$$

then (l_1, l_2, l_3) are the direction cosines of the perpendicular to the face BCD with respect to Ox, Oy, Oz; (m_1, m_2, m_3) the direction cosines of the perpendicular to the face ACD, etc.

So $\qquad l_1^2 + l_2^2 + l_3^2 = 1$, etc.,

and
$$l_1m_1 + l_2m_2 + l_3m_3 = -\cos AB,$$
$$l_1n_1 + l_2n_2 + l_3n_3 = -\cos AC,$$
$$l_1p_1 + l_2p_2 + l_3p_3 = -\cos AD, \text{ etc.}$$

If then O is the vertex of the cone

$$u\alpha^2 + v\beta^2 + w\gamma^2 + k\delta^2 + 2u_1\beta\gamma + 2v_1\gamma\alpha + 2w_1\alpha\beta + 2r\alpha\delta + 2s\beta\delta + 2t\gamma\delta = 0,$$

and if Ox, Oy, Oz are generators of this cone, then

$$ul_1^2 + vm_1^2 + \ldots + 2u_1m_1n_1 + \ldots = 0,$$
$$ul_2^2 + vm_2^2 + \ldots + 2u_1m_2n_2 + \ldots = 0,$$
$$ul_3^2 + vm_3^2 + \ldots + 2u_1m_3n_3 + \ldots = 0.$$

Adding these equations, the condition that the cone should have three mutually perpendicular generators is

$$u + v + w + k - 2u_1\cos BC - 2v_1\cos AC - 2w_1\cos AB$$
$$- 2r\cos AD - 2s\cos BD - 2t\cos CD = 0.$$

6. *The cuboidal solid.*

If $ABCD$ is a tetrahedron self-conjugate with respect to the conicoid whose equation is

$$u\alpha^2+v\beta^2+w\gamma^2+k\delta^2=0,$$

which we can write, putting $u=1/\alpha_1^2$, $v=1/\beta_1^2$, $w=1/\gamma_1^2$, $k=-1/\delta_1^2$, as

$$(\alpha/\alpha_1)^2+(\beta/\beta_1)^2+(\gamma/\gamma_1)^2-(\delta/\delta_1)^2=0,$$

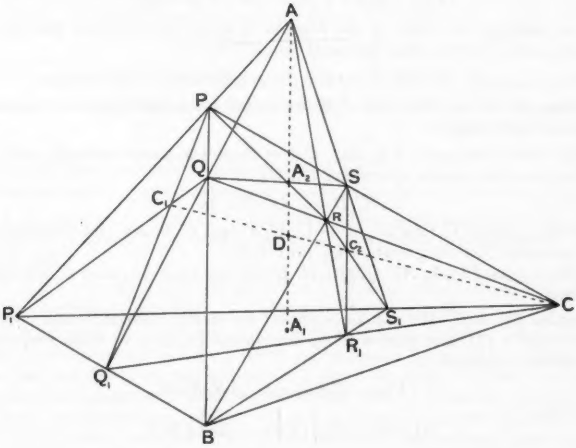
and if the tangent planes through AB, AC, BC touch the conicoid at C_1, C_2; B_1, B_2; A_1, A_2 respectively, then when $ABCD$ is taken as tetrahedron of reference it is easily seen that

the coordinates of C_1, C_2 are $(0, 0, \gamma_1, -\delta_1)$, $(0, 0, \gamma_1, \delta_1)$;

the coordinates of B_1, B_2 are $(0, \beta_1, 0, -\delta_1)$, $(0, \beta_1, 0, \delta_1)$;

the coordinates of A_1, A_2 are $(\alpha_1, 0, 0, -\delta_1)$, $(\alpha_1, 0, 0, \delta_1)$.

Therefore these pairs of points lie respectively on CD, BD, AD and divide these three edges harmonically.



It can also be shown that the six tangent planes intersect in the eight points P, Q, R, S, P_1, Q_1, R_1, S_1 whose coordinates are $(\pm\alpha_1, \pm\beta_1, \pm\gamma_1, \pm\delta_1)$, and that they generate a cuboidal solid corresponding in three dimensions to the quadrilateral in two dimensions.

The coordinates of R are $(\alpha_1, \beta_1, \gamma_1, \delta_1)$;

of P $(\alpha_1, -\beta_1, -\gamma_1, \delta_1)$;

of P_1 $(\alpha_1, \beta_1, \gamma_1, -\delta_1)$;

of Q $(\alpha_1, \beta_1, -\gamma_1, \delta_1)$;

of Q_1 $(-\alpha_1, \beta_1, -\gamma_1, \delta_1)$; and so on.

We can now prove that PQ_1, P_1Q intersect in C_1. For, the coordinates of any point on PQ_1 are

$$\alpha_1(1-\theta),\ \beta_1(-1+\theta),\ -\gamma_1(1+\theta),\ \delta_1(1+\theta),$$

and when $\theta=1$, these reduce to the coordinates of C_1 given above. Similarly C_1 lies on P_1Q.

Hence *the points of intersection of the diagonals of the faces are the points of contact of the faces with the conicoid.*

With reference to the conicoid, the octahedron formed by the six points of contact A_1, A_2; B_1, B_2; C_1, C_2 is the reciprocal of the cuboidal solid. From the figure it can be seen that P_1QS is the polar plane of P, and that PC_1, that is to say PQ_1, and P_1Q are polar lines.

Hence *the diagonals of the faces are polar lines with respect to this conicoid.*

When the conicoid is the self-polar sphere of the tetrahedron, the cuboidal solid has the following features:

(1) The sphere touches the edges of the two tetrahedra PQ_1RS_1, P_1QR_1S. Hence

$$PQ_1+RS_1=PR+Q_1S_1=PS_1+Q_1R,$$
$$P_1Q+R_1S=P_1R_1+QS=P_1S+QR_1.$$

Hence on adding, *the sum of the lengths of the diagonals of a pair of opposite faces is the same for the three pairs of faces.*

(2) *The diagonals of each of the six faces intersect at right angles.*

For example, since PQ_1 and P_1Q are polar lines with respect to the sphere they are at right angles.

(3) *The third diagonals AB, BC, CA of these six quadrilaterals each subtends a right angle at the points of contact C_1, C_2, ...*

That is, $\angle AC_1B=\angle AC_2B=\tfrac{1}{2}\pi.$

(4) Hence, since $C_1(APBQ)=-1$, C_1A and C_1B are the bisectors of the angles between the diagonals PQ_1 and P_1Q.

Thus *the angles PC_1A, AC_1Q, QC_1B, BC_1Q are each* $\tfrac{1}{4}\pi$, and similarly for the other faces.

Properties (3) and (4) may be verified by using quadriplanar coordinates. Thus, to verify (3), the power T of any point $(\alpha, \beta, \gamma, \delta)$ with respect to the sphere whose equation is

$$(\Sigma l\alpha)\,.\,(\Sigma A\alpha)-\Sigma c^2AB\alpha\beta=0$$

is given by

$$9V^2T^2=(\Sigma l\alpha)(\Sigma A\alpha)-\Sigma c^2AB\alpha\beta.$$

The equation of the self-polar sphere can be written

$$\tfrac{1}{2}\{A\alpha(b^2+c^2-a^2)+B\beta(a^2+c^2-b^2)+C\gamma(a^2+b^2-c^2)$$
$$+D\delta(e^2+f^2-a^2)\}(\Sigma A\alpha)-\Sigma c^2AB\alpha\beta=0,$$

or $\tfrac{1}{2}\Sigma A^2\alpha^2(b^2+c^2-a^2)=0.$

Hence if T_A is the power of A with respect to the self-polar sphere,

$$9V^2T_A{}^2=\tfrac{9}{2}V^2(b^2+c^2-a^2),$$

or $AC_1{}^2=\tfrac{1}{2}(b^2+c^2-a^2).$

Similarly $BC_1{}^2=\tfrac{1}{2}(a^2+c^2-b^2).$

Hence $AC_1{}^2+BC_1{}^2=c^2=AB^2.$

Thus $\angle AC_1B=\tfrac{1}{2}\pi.$

For (4), to prove that PQ_1, BC_1 intersect at an angle of $\frac{1}{4}\pi$, we proceed thus: the coordinates of P are $\lambda\alpha_1,\ -\lambda\beta_1,\ -\lambda\gamma_1,\ \lambda\delta_1$,

where $$3V/\lambda = A\alpha_1 - B\beta_1 - c\gamma_1 + D\delta_1;$$

the coordinates of Q_1 are $-\mu\alpha_1,\ \mu\beta_1,\ -\mu\gamma_1,\ \mu\delta_1$,

where $$3V/\mu = -A\alpha_1 + B\beta_1 - C\gamma_1 + D\delta_1,$$

and
$$\begin{aligned} A\alpha_1\surd(b^2+c^2-a^2) &= B\beta_1\surd(a^2+c^2-b^2) \\ &= C\gamma_1\surd(a^2+b^2-c^2) \\ &= D\delta_1\surd-(e^2+f^2-a^2). \end{aligned}$$

The direction cosines (l_1, m_1, n_1, p_1) of the line PQ_1 are proportional to
$$\alpha_1(\lambda+\mu),\ -\beta_1(\lambda+\mu),\ -\gamma_1(\lambda-\mu),\ \delta_1(\lambda-\mu),$$
or $$\alpha_1(-C\gamma_1+D\delta_1),\ \beta_1(C\gamma_1-D\delta_1),\ \gamma_1(A\alpha_1-B\beta_1),\ \delta_1(-A\alpha_1+B\beta_1).$$

The direction cosines (l_2, m_2, n_2, p_2) of BC_1 are proportional to
$$0,\ C\gamma_1 - D\delta_1,\ -B\gamma_1,\ B\delta_1.$$

Hence we can show that
$$\begin{aligned} \Sigma c^2AB(l_1m_2+l_2m_1) &= -(C\gamma_1-D\delta_1)^2/\beta_1, \\ \Sigma c^2ABl_1m_1 &= -(C\gamma_1-D\delta_1)^2, \\ \Sigma c^2ABl_2m_2 &= -(C\gamma_1-D\delta_1)^2/2\beta_1^2. \end{aligned}$$

Thus if θ be the angle between PQ_1 and BC_1, we have by the formula of § 1,
$$\cos\theta = \pm 1/\surd 2,$$
and hence the lines intersect at an angle of $\frac{1}{4}\pi$. R. T. R.

1420. I am now going to work as hard at Mathematics, harder indeed, for a few days' work at Classics is a trifle compared to a day's Mathematics. (From a letter.) (p. 432.)

1421. . . . No one questions the discipline of Mathematics though that of French and German be ever so doubtful. . . . (From a letter.) (p. 539.)

1422. I remember the scholars' table at Balliol with delight. There, I think, I found such social pleasure as I have never found beaten since. The mistake of ——, however, has not been an original want of *esprit de corps*. It had once as much as ——, but the College gave itself up body and soul to the study of mathematics. And a mere mathematical college will never stand its ground against one that takes to more human studies. Say what they will, the men of mere science are the hewers of wood and drawers of water for the men of politics and philosophy. (From a letter.) (p. 594.)

1423. Mathematics take it out of one, as the saying is, and still more so with me, because I am so fond of it that while I am on this branch of study I can hardly think of anything else. (From a letter.) (p. 657.)

1424. I know that two and three make five not only here but in the distant planet Saturn. How do I know it? The answer is plain if Space is a form in my own mind; but if it be a thing existing independently of me, no explanation can be given at all and the universality of our knowledge of the properties of space is an ultimate fact with which we begin, but for which we cannot account. (From a letter.) (p. 692.)

Gleanings 1420–4 from *Memoirs of Archbishop Frederick Temple*, II [Per Mr. A. F. Mackenzie].

MATHEMATICAL NOTES.

1636. *Numerical solution of equations.*

The second approximation to the solution of a quadratic given by Mr. R. A. Fairthorne (*Gazette*, XXVI, p. 109) is the second approximation given by the Newton-Raphson method (Whittaker and Robinson, *Calculus of Observations*, p. 85).

As the labour of using Horner's method is often mentioned (and the frequency of mistakes less often), I think it is worth while to call attention again to the fact that most of this labour and risk of error arises through the use of synthetic division. It appears to be supposed that the easiest way of forming $6+3\times 5$ is to add 5 to 6 three times ; but I doubt it. The method that I should recommend is virtually Newton's original one (cf. L. J. Mordell, *Nature*, 119, 1927, Mar. 26, Suppl. p. 42 ; Jeffreys, *Nature*, 119, 1927, p. 565).

Take the equation $\qquad x^3-2x-5=0.$

Put $x=2+x_1$. Then

$$5+2(2+x_1)-(8+12x_1+6x_1^2+x_1^3)=0,$$

or $1-10x_1-6x_1^2-x_1^3=0.$

Put $x_1=0{\cdot}1+x_2$:

$$1-10(0{\cdot}1+x_2)-6(0{\cdot}1+x_2)^2-(0{\cdot}1+x_2)^3=0.$$

Expand and simplify, then

$$-0{\cdot}061-11{\cdot}23x_2-6{\cdot}3x_2^2-x_2^3=0.$$

We can proceed one place at a time as in Horner's method, but at this stage it is easier to rearrange and proceed as follows :

$$11{\cdot}23x_2=-0{\cdot}061-6{\cdot}3x_2^2-x_2^3.$$

Put $x_2=-0{\cdot}006$ on the right ; then we get a further approximation $x_2'=-0{\cdot}00546$. Now try $x_2=-0{\cdot}0054$ and $x_2=-0{\cdot}0055$ on the right ; they give as the next approximations $x_2'=-0{\cdot}00544822$ and $x_2'=-0{\cdot}00544883$. Taking now $x_2=-0{\cdot}005448$ we get by interpolation $x_2'=-0{\cdot}00544851$, and this is unaltered by further approximation to this number of figures. Hence $x=2{\cdot}1-0{\cdot}00544851=2{\cdot}09455149$, which is right to one unit in the last place.

The method of interpolation used in the last stage will work whenever the Newton-Raphson one does, and not only for algebraic equations. It is applicable immediately on a multiplying machine, and is more convenient, I think, than the method based on inversion of series given by Mr. W. G. Bickley (*Gazette*, XXVI, 102–4).

I do not understand Bickley's remark that the series given by his method are not convergent but asymptotic. If $\zeta=f(z)$ converges in a closed region of non-zero extent in each direction about $z=z_0$ and if $f'(z)\neq 0$ in this region, then there is a closed curve about $\zeta_0=f(z_0)$ such that z is regular at all points on and within it, and z can therefore be expanded as a convergent series in $\zeta-\zeta_0$. HAROLD JEFFREYS.

1637. *Remarks on Note* 1591.

(i) Given the " flying start ", Fairthorne's formula

$$x_{n+1}=(x_n^2-c)/(2x_n+b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

for iterative approximation to a root of

$$f(x)=x^2+bx+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

converges rapidly, but his statement as to the number of reliable figures is not *generally* true. For instance, if the roots of (2) are nearly equal, the denominator of (1) is small, and then x_{n+1} may be *less* accurate than x_n.

(ii) The formula (1) is in fact the Newton approximation,

$$x_{n+1}=x_n-f(x_n)/f'(x_n).$$

(iii) In this, x_{n+1} is *certainly* better than x_n if $f(x_n)$ and $f''(x_n)$ have the same sign; here, if $f(x_n)$ is positive, *i.e.* if x_n is *not between* the roots α and β of (2).

(iv) If x_n and x_{n+1} are approximations to α, then

$$\alpha-x_{n+1}=\alpha-(x_n^2-\alpha\beta)/(2x_n-\alpha-\beta),$$

or

$$\frac{\alpha-x_{n+1}}{\alpha-x_n}=\frac{(\alpha-x_n)}{(\alpha-x_n)+(\beta-x_n)}. \qquad \text{(3)}$$

Thus x_{n+1} is a better approximation than x_n if $(\alpha-x_n)$ and $(\beta-x_n)$ have the same sign. This repeats the conclusion reached in (iii).

(v) If x_n lies between the roots, x_{n+1} is a better approximation only if $|\beta-x_n|>2|\alpha-x_n|$, *i.e.* if x_n lies outside the middle third of the interval (α, β).

(vi) Returning to (3), it is clear that if x_n is outside the interval (α, β), then $(\alpha-x_{n+1})$ has the sign of $(\alpha-x_n)$ and $(\beta-x_n)$; x_{n+1} (and so x_{n+r}) is outside, and on the same side of, the interval as x_n.

(vii) If x_n is within, and nearer α than β, then x_{n+1} is outside on the α-side; in this case $x_{n+r}\to\alpha$. But if x_n is nearer to β than to α, then $x_{n+r}\to\beta$.

(viii) Finally, if x_n is outside, on the α-side,

$$\left|\frac{\alpha-x_{n+1}}{\alpha-x_n}\right|<\left|\frac{\alpha-x_n}{\beta-\alpha}\right|.$$

This provides the scale of magnitude—that of $\beta-\alpha$—by which " flying " and " crawling " starts can be distinguished. It also indicates that no general statement, independent of $(\beta-\alpha)$, can be made regarding the rapidity of convergence.

W. G. B.

1638. *On Note* 1591.

An interesting formula for the roots of a quadratic is the following:

If x_n is an approximation to a root of

$$x^2+bx+c=0,$$

then a better approximation is given by

$$x_{n+1}=x_n\left\{\frac{2(x_n^2-c)-f(x_n)}{2(x_n^2-c)+f(x_n)}\right\}.$$

This is in general a better approximation than that of Note 1591.

When c is negative, the error in the above formula is in defect by

$$h^2(1+b)/\{2(x_n^2-c)+f(x_n)\},$$

and the error in the approximation

$$x_{n+1}=\tfrac{1}{2}(x_n^2-c)/(x_n+\tfrac{1}{2}b)$$

is in excess by

$$h^2/(2x_n+b).$$

Thus in this case the errors have opposite signs and a combination of the two formulae is possible. The result of combining them is

$$x_{n+1}=\frac{x_n(x_n^2-3c)-bc}{(3x_n^2-c)+b(3x_n+b)}$$

with an error $\qquad h^3/\{(3x_n^2-c)+b(3x_n+b)\}$. R. H. BIRCH.

1639. *On Note* 1591.

Care should be observed in the use of the approximation given in Note 1591 in the case when the two roots are nearly equal.

For example, $x_n=20$ is an approximation correct to two figures to a root of the equation

$$x^2-40\cdot3x+406\cdot02=0.$$

The formula of Note 1591 gives $x_{n+1}=20\cdot07$ as an approximation correct to four figures, but the roots are 20·1 and 20·2 exactly.

A more striking example is supplied by the equation

$$x^2-6\cdot39999x+10\cdot23996699=0\,;$$

here $x_n=3\cdot20$ leads to $x_{n+1}=3\cdot30100$, but the roots are 3·201 and 3·19899.

Let the roots of the equation

$$x^2+bx+c=0$$

be α and β; let x_n be an approximation to α correct to n figures. Then

$$|\,\alpha-x_n\,|<\tfrac{1}{2}\,.\,10^{a-n+1}$$

where a is a positive or negative integer or zero, such that

$$10^a\leqslant x_n<10^{a+1}.$$

Then

$$\begin{aligned}|\,\alpha-x_{n+1}\,|&=(\alpha-x_n)^2/|\,2x_n+b\,|\\&<10^{2a-2n+2}/4\,|\,2x_n+b\,|.\end{aligned}$$

For x_{n+1} to be an approximation to α correct to $2n$ figures,

$$|\,\alpha-x_{n+1}\,|<\tfrac{1}{2}\,.\,10^{a-2n+1}.$$

A sufficient condition for this is

$$10^{-a-1}\,.\,2\,|\,2x_n+b\,|\geqslant1,$$

that is,

$$2\,|\,2x_n+b\,|\geqslant10^{a+1}.$$

To this order of magnitude, $x_n=\alpha$, and therefore since $b=-(\alpha+\beta)$, this condition becomes

$$2\,|\,\alpha-\beta\,|\geqslant10^{a+1}$$

or

$$4(b^2-4c)\geqslant10^{2(a+1)}.$$

That the condition is not necessary can be shown by considering the equation

$$x^2-6\cdot001x+8\cdot004=0\,;$$

$x_n=2\cdot00$ is an approximation correct to three figures to a root. The formula gives $x_{n+1}=2\cdot00100$ as the approximation to six figures and this is correct, although $4(b^2-4c)$ is less than $10^{2(a+1)}$. JOHN DEANS.

1640. *Extension and application of a determinant formula.*

A brief description of a method for simplifying the laborious work of computing earthwork quantities, which is of almost daily occurrence in the life of a civil engineer, may be of interest to some readers of the *Gazette* as a possible illustration for class work.

The area of a triangle expressed in the form of a determinant is

$$A = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{.......................................(i)}$$

which can be written in the alternative form

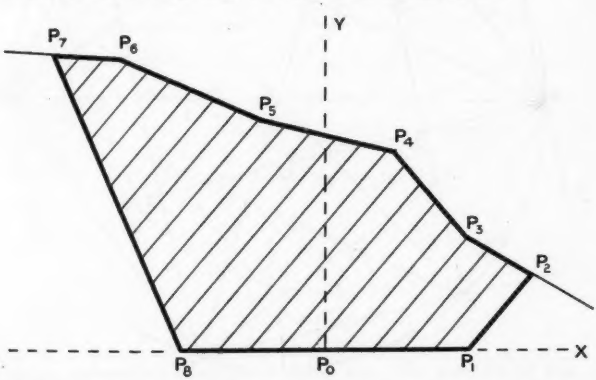
$$A = \tfrac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{bmatrix} \quad \text{...........................(ii)}$$

where the multiplications indicated by the full arrows are preceded by a plus sign, and those indicated by the dotted arrows by a minus sign.

Now it is easy to prove that if we have a plane rectilinear figure with n vertices, the area is given by

$$A = \tfrac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \cdots x_n & x_1 \\ y_1 & y_2 & y_3 \cdots y_n & y_1 \end{bmatrix}. \quad \text{....................(iii)}$$

To illustrate the practical application of this expression, let us consider a typical cross-section of a railway cutting, as in the diagram.



The first step in the preparation of the estimate of the cost of construction of a railway is the computation of the quantity of earthwork involved, for which areas of cross-sections are required every 50 or 100 feet along the line.

The only way to compute these areas of cross-section hitherto known to me is to split the cross-section up into convenient rectangular and triangular areas and obtain the area of each sub-division and add, or else to use the planimeter; both methods involve very considerable labour. But if the points $P_1, P_2, \dots P_n$ shown on the above cross-section are assigned coordinates, in the form of distances from the centre line and heights above the bed of the cutting, the area can be written down from the engineer's survey book, in the form of expression (iii), due care being taken with the signs of the coordinates; the remaining arithmetic work is simple and straightforward and may be performed by subordinate staff.

I am surprised that this simplification has not been used before, but reference to several recent books on surveying and to recent issues of Civil Engineers' Handbooks shows no advance on the old methods which I learned in 1912.

F. F. FERGUSON.

1641. *The Formation of Integral Cyclic Hexagons* (see Notes 1514 and 1557).

Two such hexagons may be obtained from any two Pythagorean sets (a, b, c) and (x, y, z), in which $a^2+b^2=c^2$ and $x^2+y^2=z^2$. Let each of these sets be multiplied by the third member of the other. We then have

$$(az)^2+(bz)^2=(cz)^2 \text{ and } (cx)^2+(cy)^2=(cz)^2,$$

equations giving two right-angled triangles with the same hypotenuse.

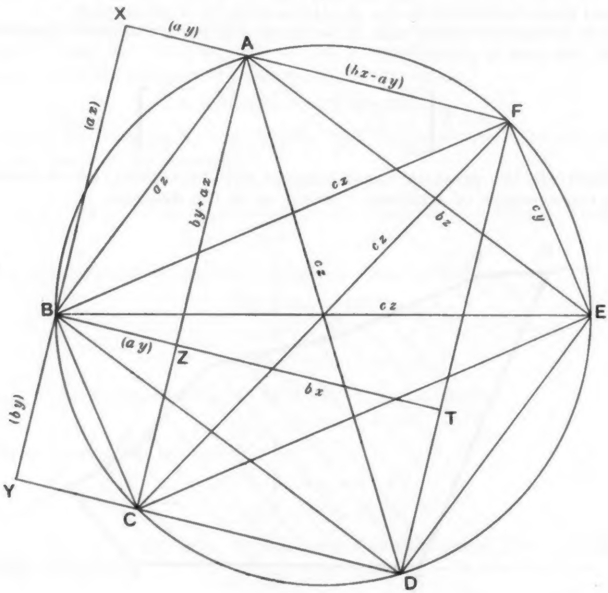


FIG. 1.

As in Fig. 1, let the diameter BE of a circle be of length cz, and ABE and FBE two inscribed right-angled triangles such that $BA=az$, $AE=bz$, $BF=cx$, $FE=cy$. Then, by Ptolemy's Theorem, $AF=bx-ay$. If FC is the diameter through F, we have

$$AC^2=(cz)^2-(bx-ay)^2=(a^2+b^2)(x^2+y^2)-(bx-ay)^2=(ax+by)^2$$

and therefore $AC=ax+by$. The rest of the hexagon follows at once.

If perpendiculars are drawn from B to AF, CD, AC, FD, it follows from the theorem $BA.BF=BX.BE$ that $BX=ax$, $BY=by$, $BZ=XA=ay$, $BT=XF=bx$.

The hexagon of Note 1514 is the simplest possible case, and, viewed from the East, is formed by the sets (3, 4, 5) and (4, 3, 5). That is, $a=y=3$,

$b = x = 4$, $c = z = 5$. Since the two sets are variations of each other, the two cases obtainable are identical.

The hexagon of Note 1557, viewed from the South-East, is given by the sets (5, 12, 13) and (4, 3, 5). A distinct hexagon, shown in Fig. 2, is given by the sets (4, 3, 5) and (12, 5, 13). Any further combinations of these two sets will give mirror images of the two hexagons already found.

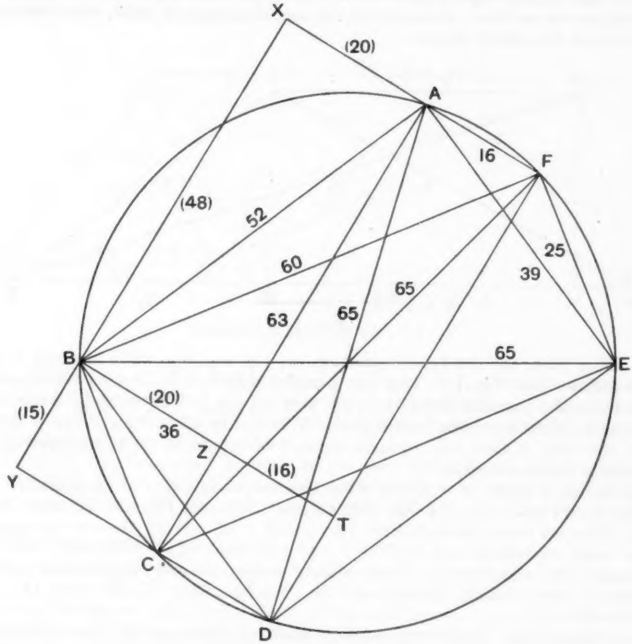


FIG. 2.

In Fig. 2, if drawn on a small scale, AD, BT, and CE appear to concur, but it is an interesting exercise to show that the product of the ratios in which these lines cut the sides of triangle BED, taken clockwise, is not unity but 704/625, and in the general case

$$a(abz^2 - xyc^2)/bc^2y^2.$$

Two more hexagons are given by the sets (3, 4, 5) and (15, 8, 17), and by the sets (4, 3, 5) and (15, 8, 17).

Any number of hexagons can be thus formed, but the above five appear to be the only hexagons of this nature with all integers less than 100.

SYDNEY THOMSON.

1642. *The Formation of Integral Trapeziums* (see Note 1560).

By an integral trapezium we mean a trapezium which has integral sides, diagonals, altitude and area. The trapezium of Note 1560 is a special case of this.

Several sets of integral solutions can be found for the equation $x^2 - y^2 = a^2$, where a is a given integer. If $a^2 = bc$, where b and c are both odd or both even, and $b > c$, then $x = \frac{1}{2}(b+c)$ and $y = \frac{1}{2}(b-c)$. There is one set (x, y) for each way in which a^2 can be expressed as the product of two integers.

Now if four values of y can be found for a given a, such that $y_p + y_q = y_r \pm y_s$, an integral trapezium can be found for each set of four values of y. If the positive sign on the right-hand side is used, the base angles of the trapezium are both acute (or both obtuse); if the negative sign is used, one base angle is acute and the other obtuse.

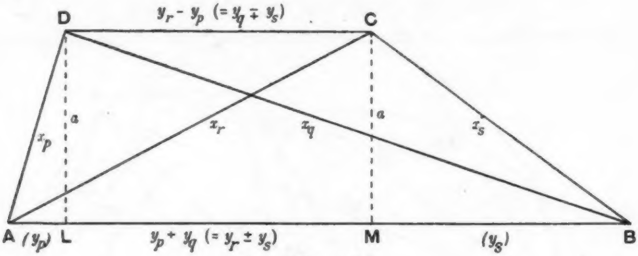


Fig. 1

In either case, for the basic equation $y_p + y_q = y_r \pm y_s$, the integral lengths are as follows (see Fig. 1). The non-parallel sides are x_p and x_s, the diagonals x_q and x_r, the parallel sides $y_r - y_p$ and $y_p + y_q$. The area is $\frac{1}{2}a(y_r + y_q)$. When a is even, the area is integral. When a is odd, then every y is even (since the sum of two odd squares cannot be equal to an even square), and the area is again integral.

The smallest value of a which gives rise to enough sets of (x, y) is 24. The sets are then (145, 143), (74, 70), (51, 45), (40, 32), (30, 18), (26, 10) and (25, 7). From these we have the equality $70 + 7 = 45 + 32$, giving the two trapeziums whose basic equations are (1) $70 + 7 = 45 + 32$ and (2) $7 + 70 = 32 + 45$. Also we have $70 = 45 + 18 + 7$, from which three distinct trapeziums can be obtained, with basic equations $18 + 7 = 70 - 45$, $7 + 45 = 70 - 18$, and $45 + 18 = 70 - 7$.

Two trapeziums are shown, and in each of them one of the triangles of the trapezium has an integral altitude other than a. This happens when $a(y_r - y_p)$ or $a(y_p + y_q)$ is divisible by one of the x's. There does not appear to be any rule governing this possibility.

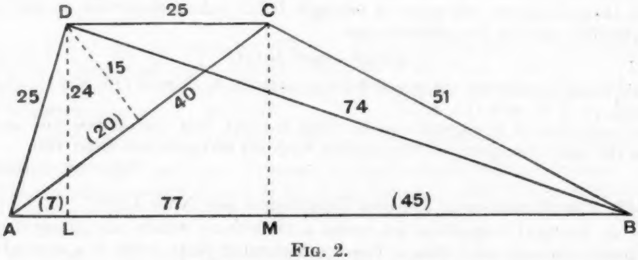


Fig. 2.

Basic equation: $7 + 70 = 32 + 45$.

When $a=36$, no basic equations are possible, and the next value of a which gives suitable values for (x, y) is 40. The sets (104, 96), (85, 75), (50, 30), (41, 9) are connected by $30+75=96+9$, giving the trapezium of Note 1560 with basic equation $30+75=96+9$, and its companion, shown in Fig. 2, with basic equation $30+75=9+96$. The companion has no integral altitude other than 40.

The smallest odd value of a giving rise to a basic equation is 45, for which $336=200+108+28$, leading to three trapeziums.

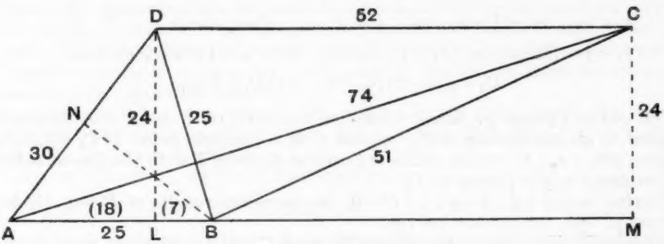


FIG. 3.

Basic equation : $18+7=70-45$.

The trapezium of Fig. 3 fulfils the condition of Note 1560, that no two sides should be equal, and that the perpendicular distance $B(DA)$ should be integral, and therefore provides a solution with integers smaller than those of the solution given in Note 1560.

In Fig. 3 the diagonals appear to be perpendicular to each other, but investigation shows that the angle between them is arc sin (924/925), the acute angles being subtended by the parallel sides. SYDNEY THOMSON.

1643. *On generalised inversion.*

Suppose that any line through a fixed point O meets three algebraic curves $\Gamma, \Gamma_1, \Gamma_2$ in P, P_1, P_2 and that $OP_1 . OP_2 = OP^2$ for all positions of the line. Let n be the degree of Γ, m the class, D the deficiency, δ, κ, τ and ι the number of nodes, cusps, bitangents and inflexions respectively. Let $n_1, m_1, \ldots$ and $n_2, m_2, \ldots$ be the similar quantities for Γ_1 and Γ_2.

If Γ is a circle with centre O, Γ_1 and Γ_2 are inverse curves with respect to Γ. If Γ is a straight line, we have the case discussed in detail by F. H. Hummel in *Math. Gazette*, XXVI (1942), p. 76. It is a particular case of a well-known quadratic transformation; see, for instance, Hilton's *Plane Algebraic Curves* (Clarendon Press, 1932), p. 130.

The following more general results are readily proved. It is assumed that Γ and Γ_1 have general positions, *e.g.* they do not go through O, touch the line at infinity, etc.

The Plücker's numbers of Γ_2 are given by $n_2=2nn_1$, $m_2=mn_1+nm_1+2nn_1$, $\kappa_2=n\kappa_1+\kappa n_1$.

For instance, we deduce

$$\delta_2=\tfrac{1}{2}nn_1(4nn_1-n-n_1-2)+(n\delta_1+\delta n_1),\quad \iota_2=6nn_1+n\iota_1+\iota n_1,$$

$$D_2=(n-1)(n_1-1)+(nD_1+Dn_1).$$

The curve Γ_2 has n linear branches touching at O each line through O parallel to an asymptote of Γ_1, so that O is an nn_1-ple point of Γ_2 (multiple

point of order nn_1) equivalent to $\frac{1}{2}nn_1(nn_1+n-2)$ nodes. The curve has also n_1 linear branches touching the line at infinity at each infinite point of Γ, so that each such point is equivalent to $n_1(n_1-1)$ nodes of Γ_2. The nodes of Γ_1 and Γ account for $n\delta_1$ and δn_1 nodes of Γ_2 respectively, and Γ_2 has also $\frac{1}{2}nn_1(n-1)(3n_1-2)$ other nodes each due to some line through O making OP^2/OP_1 the same for two different intersections of Γ or Γ_1 with the line.

These results want modification, if Γ and Γ_1 have specialised positions. As an illustration we may consider the case in which Γ is a curve of even order n having O as a centre of symmetry. Then

$$n_2=nn_1,\quad m_2=\tfrac{1}{2}(nm_1+mn_1+nn_1),\quad \kappa_2=\tfrac{1}{2}(n\kappa_1+\kappa n_1),$$
$$\delta_2=\tfrac{1}{4}nn_1(2nn_1-n-n_1-1)+\tfrac{1}{2}(n\delta_1+\delta n_1),\quad \iota_2=\tfrac{1}{2}(3nn_1+n\iota_1+\iota n_1),$$
$$D_2=\tfrac{1}{4}(n-2)(n_1-2)+\tfrac{1}{2}(nD_1+Dn_1).$$

The curve Γ_2 has $\frac{1}{2}n$ linear branches touching at O each line through O parallel to an asymptote of Γ_1, so that O is a $\frac{1}{2}nn_1$-ple point of Γ_2 equivalent to $\frac{1}{8}nn_1(nn_1+n-4)$ nodes, while the intersections of Γ with the line at infinity are ordinary n_1-ple points of Γ_2.

Putting $n=m=2$, $\delta=\kappa=\iota=D=0$, we have the results of *Plane Algebraic Curves*, p. 165, for inverse curves.

A formula which holds for all positions of Γ and Γ_1, whether specialised or not, is

$$\cot\phi_1+\cot\phi_2=2\cot\phi,$$

where ϕ, ϕ_1, ϕ_2 are the angles made with OP_1PP_2 by the tangents at P, P_1, P_2 to the curves Γ, Γ_1, Γ_2. HAROLD SIMPSON.

1644. *Tests for divisibility.*

For some small integers there exist rules to determine whether any large integer contains them as factors. The rules for 9 (including its factor 3), 11, 2 and 5 are well known.* In every case the problem of finding whether a given number contains 9 (or 11, 2 or 5, as the case may be) as a factor is reduced to a problem of the same type, where instead of the given number another smaller one appears.

We proceed to determine the scale of reduction. If M is the number whose factors, if any, are sought, and p the factor, then the problem is to find if an integer n exists such that $np=M$. We shall use the notation adopted by Hardy and Wright (*loc. cit.*) and write $p \mid M$ to express the fact that M is divisible by p. If we know a rule for p of the kind referred to above, the problem is reduced to the form $p \mid m$ where $m<M$. If $p=2$ or 5, m is independent of M; in this case $0 \leqslant m < 10$. In the case $p=9$ the reduction is $0<m\leqslant 9n$ where n is the number of digits in M. The inequality is similar for $p=11$, namely, $0\leqslant m\leqslant 9(n+1)/2$.

Since the number of digits in M is of the same order of magnitude as $\log_{10} M$, we may speak of a logarithmic reduction for 9 and 11.

A rule for 7, 11 and 13 is also mentioned in the book cited above.† This rule depends on the fact that $7.11.13=1001$. The reduction is

$$0\leqslant m\leqslant M-1001.$$

In fact, $m=M-1001.\ A_{n-1}.10^{n-4}$ where A_{n-1} is the value of the highest decimal place of M. This reduction is so slight in all cases of practical importance that the method has to be applied several times in succession to

* They are explained in G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers* (Oxford 1938), Ch. 9, § 5.

† Hardy and Wright refer to Grunert, *Archiv* . . . (1864).

be of use. The lowest number to which m can be reduced by an $(n-3)$-fold application of the method has still three digits.

A different method can be suggested as follows. If s is an integer such that $p \mid 10s+1$, and

$$M = A_{n-1}10^{n-1} + A_{n-2}10^{n-2} + \dots + A_0,$$

then $$p \mid A_{n-1} \,.\, 10^{n-1} + A_{n-2} \,.\, 10^{n-2} + \dots + A_0, \qquad \text{(i)}$$

provided that $$p \mid A_{n-1} \,.\, 10^{n-2} + A_{n-2} \,.\, 10^{n-3} + \dots + A_1 - sA_0. \qquad \text{(ii)}$$

This gives approximately the same reduction as the former method, namely $0 \leqslant m \leqslant \frac{1}{10}M$, but it may be carried out mentally with greater ease and by an $(n-1)$-fold application m may be compressed to one digit. If, however, one negative number occurs, we have to apply the rule n times to reduce m to one digit (see example below).

As an example we may take 7, for which $s=2$; if we apply the rule to 285964 we have in succession 285964, 28588, 2842, 280, 28, -14, $+7$. This shows that 285964 is divisible by 7.

To prove the result, we have, by hypothesis,

$$p \mid A_{n-1} \,.\, 10^{n-2} + A_{n-2} \,.\, 10^{n-3} + \dots + A_1 - sA_0. \qquad \text{(ii)}$$

If $b \mid a$ and $b \mid c$, where a, b, c are three integers appropriately chosen, it follows that $b \mid ka + lc$ for any integral values of k and l. So, if we multiply (ii) by 10 and add $(10s+1)A_0$, which of course is divisible by p, by hypothesis, we obtain

$$p \mid A_{n-1} \,.\, 10^{n-1} + A_{n-2} \,.\, 10^{n-2} + \dots + A_0. \qquad \text{(i)}$$

Q.E.D.

A method of this kind exists for all values of p which are not multiples of the factors of the base, that is, 2 and 5 in the decimal system. Should it be required to find whether a given number is divisible by the power of a prime, for definiteness say 13^2, the test for 13 is applied first; if the result is positive, the number can be divided by 13 and the test can be applied to the quotient. This can be repeated for any power higher than the second.

Thus the method is shown to be sufficient for any investigation, since we need only test for divisibility by prime numbers. For 2 or 5 a different method is employed (see above). It will be noticed that from this aspect the well-known rule for 11 is only a special case of the above. PAUL COHN.

1645. *Addition to Note* 1613.

The method of bisecting a segment, and of finding the centre of a circle with the compass alone, is given in the recently published book *What is Mathematics?* by Courant and Robbins, p. 145. The constructions are substantially the same as those given above, but it is interesting to note that they both depend on a geometrical construction of inverse points wo a circle. In the construction for bisecting a line, the mid-point M required is the inverse of G wo the circle S_1. In the problem of finding the centre of a given circle, the centre required is the inverse of D wo the circle of centre A and radius a. (There is no need for the radius a to be greater than the radius of the given circle as stated above.) E. G. PHILLIPS.

1425. "You only did what nine hundred thousand nine hundred and ninety-nine women out of every million would have done." He spoke with a lightness that surprised himself. Once he had thought that Madeleine was the one woman in a million.—Herman Landon, *The Picaroon does Justice* (Cassell), 1929, p. 164. [Per Mr. G. E. Strawson.]

CORRESPONDENCE.

SPELLING.

To the Editor of the *Mathematical Gazette.*

SIR,—In a School Certificate script I once had a gem which should be added to Dr. Maxwell's list. It was " hysoseles ".

Yours, etc., BERTHA JEFFREYS.

Girton College, Cambridge.

EXAMINATION QUESTIONS.

To the Editor of the *Mathematical Gazette.*

SIR,—The question quoted by Mr. Newling from the Tripos of 1894 is not altogether contemptible, for it affords an excellent exercise in the art of presenting a proof economically in a form independent of the accidents of a figure. In the argument

$$CE/YF = EA/YF = EX/AF = EX/FB$$

each ratio is algebraic, that is, bears a significant sign.

Mr. Newling has by no means found a record low level. My own entry for this competition is from a London M.Sc. examination paper, 1937, and I back it against all comers :

" Establish the identity

$$\Sigma \frac{e_\alpha - e_\beta}{\sqrt{(\wp u - e_\alpha)} - \sqrt{(\wp u - e_\beta)}} = -2\{\sqrt{(\wp u - e_1)} + \sqrt{(\wp u - e_2)} + \sqrt{(\wp u - e_3)}\}$$

where on the left-hand side the three differences are taken in the cyclic order (1 2 3)."

Yours, etc., E. H. NEVILLE.

SIR,—Would the Tripos examiners of 1894 expect one of the following arguments :

(i) Take a series of positions of AXY. Then $(X\ldots) = (Y\ldots)$ and D is a common point of both ranges. Hence $B(Y\ldots) = C(X\ldots)$ and BDC is a common ray. Hence the locus of the intersection of BY and CX is a straight line, which, by taking two special cases of AXY, namely AB and AC, is the line at infinity.

(ii) Apply Pappus' theorem to the two triads of collinear points

$$\begin{array}{ccc} 1 & 2 & 3 \\ A & X & Y \\ D & B & C \end{array}$$

$\begin{pmatrix} AB \\ DX \end{pmatrix} \begin{pmatrix} AC \\ DY \end{pmatrix} \begin{pmatrix} BY \\ CX \end{pmatrix}$ are collinear ; that is, BY, CX intersect on the line at infinity.

(iii) Apply the reciprocal of Pappus to the two triads of concurrent lines

$$\begin{array}{ccc} 1 & 2 & 3 \\ AB & AC & AX \\ DF & DE & DC \end{array}$$

Then BY, CX and the line at infinity are concurrent.

Yours, etc., L. SADLER.

SIR,—Mr. Newling's question (*Gazette*, p. 191) could best be answered by Mr. A. N. Whitehead, who was one of the examiners in the Tripos of 1894. If the same question were set in 1944, the candidates would probably say

" $ACXDYB$ is a hexagon inscribed in a line-pair ", or " CX and BY form homographic pencils with a common ray and two pairs of corresponding rays which meet at infinity ". Or, as analytical methods would not be forbidden by the regulations, they might resort to areal coordinates.

Yours, etc.,

A. Robson.

Sir,—Scholarship candidates given Mr. Newling's 1894 Tripos question solved it by means of similarity, overlooking the construction which gave his neat proof by congruence.

I suggest that the examiners *expected* the following (longer) solution. I use Mr. Newling's notation, but produce BY to cut AC at P and draw AZ parallel to BY cutting BC at Z; also AXY cuts BC at O. Then

$$(AXOY)=D(AXOY)=D(AECY)=-1,$$

for $AE=EC$ and DY is parallel to AC;

$$(ZCOB)=A(ZCOB)=A(ZPYB)=-1,$$

for $BY=YP$ and AZ is parallel to BY. From these it follows that CX is parallel to AZ and BY.

Yours, etc.,

H. V. Styler.

PARTIAL FRACTIONS.

To the Editor of the *Mathematical Gazette*.

Sir,—Teachers may be interested in points which arise from time to time in examination answers, as a guide to possible misunderstanding by their pupils. I have recently found a rather large number of candidates who attempted to evaluate an integral of the form

$$\int \frac{dx}{(a+bx^2)\sqrt{(c+dx^2)}}$$

by the step $$\frac{1}{(a+bx^2)\sqrt{(c+dx^2)}} \equiv \frac{Ax+B}{a+bx^2}+\frac{C}{\sqrt{(c+dx^2)}}$$

with variants of the actual form of " partial fractions ". By various devices the coefficients A, B, C were calculated.

I am writing because it seems to me that such work implies a fundamental misunderstanding of partial fractions themselves. The mechanical calculations are effected (a conference on really tidy methods would help examiners—and candidates!) but it is possible that many candidates do not fully understand just *why* their steps are legitimate.

Of course it is possible to perform mathematical calculations without fully understanding all the theory, as, say, in logarithms. But here positive dangers arise, and a treatment of the subject which excludes these seems desirable.

Yours, etc., E. A. Maxwell.

Queens' College, Cambridge.

STARRED QUESTIONS.

To the Editor of the *Mathematical Gazette*.

Sir,—The question of what makes a good scholarship question is an interesting one on which a great variety of opinion must be held by your readers. It would be valuable to have views from university teachers as well as school teachers.

Mr. Durell's example and Mr. Robson's two have the good quality of being off the beaten track, but whereas Mr. Durell's will yield to several sensible attacks, Mr. Robson's two (as far as I can see) have the defect (?) of possessing only one vulnerable point. Unless the candidate regards the triad in (i) or the fraction in (ii) as representing some geometrical entity he is unlikely to get a solution. I maintain that this would not occur to a sensible candidate until he had tried a more normal method of approach. He is therefore bound to *waste* time over questions so heavily disguised, and the more thorough and persevering he is the more time he will waste. Such questions are more suitable for trying over the week-end than in competitive examinations.

There is no harm in conundrums if they are obviously conundrums; for example, " Prove that the arithmetic mean of all the integers less than and prime to n is $\frac{1}{2}n$ " (Cambridge), but they must be original.

I would put in a plea for the type of question which tests powers of generalisation: for example, " A small ring R can slide over a smooth horizontal table, and to it are tied three strings which pass through holes A, B, C in the table and at their free ends carry weights W. Show that, if one angle of the triangle, say A, equals or exceeds 120°, R cannot rest except possibly at A, but that otherwise R can rest at the point at which BC, CA, AB each subtend an angle of 120°. Show that this point is unique and that it gives a least value to the sum $AR+BR+CR$ ". This question set in the Merton Group, March 1942, is good as far as it goes. But it could with advantage go further: " Give a construction for finding a point P at which the sum $u\,.\,AP+v\,.\,BP+w\,.\,CP$ is least " or " at which $AP+BP+CP+DP$ is least, where D is a fourth coplanar fixed point ", with a note that the question was long and carried higher marks.

Yours, etc., R. C. Lyness.

P.S. If my cap does not fit Mr. Robson's questions, I apologise to him, but it is worth flourishing as it certainly does fit far too many scholarship questions.

ERRORS IN MATHEMATICAL TABLES.

To the Editor of the *Mathematical Gazette*.

Sir,—For many years I have acted as an unofficial clearing-house on the subject of errors in mathematical tables. I have checked and inter-compared many tables, and noted all lists of errors published by others. I am now renewing my efforts in this direction, to provide material not only for a forthcoming American Quarterly on Mathematical Tables and Aids to Computation, of which more news will be given later in this *Gazette*, but also for a book entitled *A Computer's Guide to Mathematical Tables*, which I am now preparing for post-war printing.

I should be very glad to be informed of any known errors in tables, or to have references that might otherwise be overlooked to lists of errors. Where possible, the date or edition referred to should be quoted, as errors in early editions often do not appear in later editions. News of unpublished tables would also be appreciated.

There is still much useful work that can be done in checking tables, especially by inter-comparisons and by differencing. If any reader of this note would like to volunteer for such work, it will be provided, and full credit given for all results obtained.

Yours, etc., L. J. Comrie.

Scientific Computing Service Ltd.
23 Bedford Square, London, W.C. 1.

REVIEWS.

The Methodology of Pierre Duhem. By ARMAND LOWINGER. Pp. 184. 15s. 6d. 1941. (Columbia University Press, New York ; Humphrey Milford)

Pierre Duhem, who was professor of theoretical physics at Bordeaux from 1895 to his death in 1916, was one of the most interesting figures of his day : a pioneer in applied thermodynamics, a historian of science, a critic and controversialist, and a philosopher. His chief physical investigations were concerned with the phase law, chemical equilibrium, and various problems in which dynamical and thermodynamical effects are interrelated. On the history of ancient and mediaeval science he was an authority of the first rank : the long chapter on Aristotle's Physics in the first volume of his work *Le système du monde : histoire des doctrines cosmologiques de Platon à Copernic* is a masterpiece : and to him is due the credit of having first shown that at the University of Paris in the fourteenth century there flourished a group of scholastic philosophers who, rejecting the peripatetic doctrines of dynamics and the planetary motions, arrived at correct ideas regarding inertia and impetus, discovered the law connecting space and time in uniformly accelerated motion, and taught that a system is in equilibrium when the height of its centre of gravity is a minimum. Their principles, though never accepted by the general body of scholastics, were not forgotten and were known to Leonardo da Vinci and Galileo, by whom they were developed into the form inherited by Newton.

It is probable that the researches in mediaeval history will prove to be Duhem's most lasting title to fame. His essays in criticism and philosophy were less meritorious, indeed much of them of no merit at all. He strangely undervalued two of the greatest men of science the world has ever seen, Maxwell and William Thomson (Kelvin), and wrote copiously about their supposed errors : and this brings us to the book under review, for the head and front of their offending was that they broke the rules laid down for physical research by Duhem in his methodology.

The methodology, which was expounded chiefly in his book *La théorie physique, son objet et sa structure* (Paris, 1906) was essentially a delimitation of the boundary between physics and metaphysics. Duhem did not despise metaphysics as a positivist might have done : on the contrary, he regarded metaphysics and theology as the noblest of human studies : but he held that physics is a completely autonomous science, which should be kept pure from any admixture of metaphysical elements. Theoretical physics should have for its aim to codify the results of experiment by a system of mathematical laws. The interconnection of these laws should be studied, in order to secure intellectual economy, *i.e.* to reduce as far as possible the number of independent assertions from which all the phenomena may be deduced. But there should be no attempt to go behind the laws by postulating hidden mechanisms and hypothetical entities—aethers and what not, which are essentially unobservable—in order to account for them. Duhem's doctrine is in fact much the same as Newton's *hypotheses non fingo*.

Now, as is well known, William Thomson and Maxwell—and indeed MacCullagh and FitzGerald and J. J. Thomson and practically all our countrymen of the nineteenth century—violated these principles most flagrantly. William Thomson in 1853 showed that the mathematical expression for the energy stored in connection with a magnet, or with a circuit in which an electric current is flowing, may be transformed into a volume-integral extended over the whole of space : and on the basis of this result he suggested that the energy may be not localised in the substance of the circuits and magnets,

but may be distributed throughout the surrounding medium, with a density at any point proportional to the square of the magnetic force at the point. Such a hypothesis, being incapable of verification by experiment, would of course be condemned by Duhem's methodology as an intrusion of metaphysics into physics: actually it was a brilliant example of Thomson's intuitive genius: it was the starting-point of Maxwell's researches on action in the space surrounding electrified bodies, which led ultimately to his proof of the existence of electromagnetic waves and their identification with light. These investigations of Maxwell's and particularly those connected with the introduction of the displacement-current, were further offences against the methodology: and Duhem, who as early as 1893 in an article *L'École anglaise et les théories physiques* had expressed his disapproval of our national ways, published in 1902 a volume of 228 pages attacking the foundations of Maxwellian theory.

Why the volume now under review should have been written is not obvious; surely it would have been more charitable to leave the methodology in oblivion and, if one must write about Duhem, to draw attention to his achievements in mediaeval history or in chemical thermodynamics. One might speculate as to whether Mr. Lowinger's book was originally a Ph.D. dissertation: it has very much the form and character of one: and lest this remark should be thought disparaging, one may add that the work is most competently performed: the author, who has an excellent style and an evident capacity for arrangement, is scrupulously accurate and gives chapter and verse for all his assertions: moreover, his comments in the last chapter are thoroughly sensible. If the task had to be done, it could not have been done better. E. T. W.

Introduction to Logic and the Methodology of deductive sciences. By Alfred Tarski. (Enlarged and revised edition.) Pp. xviii, 239. 14s. 1941. (Oxford University Press; New York)

This is an excellent book. It will satisfy the needs of two classes of people: those who wish to know how recent developments in mathematical logic have extended the domain of mathematics and have enabled its foundations to be more securely established and its methods perfected, and those who have felt the need of a suitable textbook for a university course in mathematical logic. The aim which Dr. Tarski set for himself could not have been more admirably fulfilled. This English translation is a very much enlarged and considerably revised edition of a book originally published, in Polish, in 1936, and exactly translated into German (under the title *Einführung in die Mathematische Logik und in die Methodologie der Mathematik*) in 1937. The present translation, by Dr. Olaf Helmer, is admirably done. The printing is clear and the format altogether satisfactory.

Dr. Tarski has succeeded in giving an exposition of the fundamental notions and techniques of mathematical logic which is at once rigorously exact and easy to understand. Part I contains a general introduction to mathematical logic and the methodology of deductive sciences. The use of constants and variables is explained so clearly that the most elementary students should have no difficulty in obtaining accurate ideas on this topic; the sentential calculus, the use of implication in mathematics, truth functions and truth tables are expounded in such a way as to incite the interest of any one who has a taste for logic. The discussion of symbolism is excellent, being, in the opinion of the present reviewer, free from all the faults that usually beset this subject. The additions here to the German edition constitute a great advance in clarity. The theory of classes and the theory of relations are briefly but adequately explained. Part II, entitled "Applications of Logic

and Methodology in constructing mathematical theories ", shows, by means of a concrete example, how logic and methodology are applied in the construction of mathematical theories. The formulations of problems and conclusions are precise, rigorous, and elegant. Very little use is made of special logical notations in the main body of the work, but there is sufficient to familiarise the student with the elements of symbolism and to afford him practice in the use of it. There are numerous exercises, carefully graded, which will be of the greatest use to both students and teachers.

In the preface to this edition Dr. Tarski remarks that " the course of historical events " has taken " the most eminent representatives of contemporary logic " to the United States, and that conditions favourable to the development of logic have thus been created. He observes that the possibility of this development depends upon political and social relations not within the control of scholars, but he professes a belief that " logic leads to the possibility of better understanding between those who have the will to do so ", and, further, " by perfecting and sharpening the tools of thought, it makes men more critical—and thus makes less likely their being misled by all the pseudo-reasonings to which they are in various parts of the world incessantly exposed today ". The latter belief may reasonably be accepted ; the former does not, perhaps, sufficiently take note of the lack of the will to understand. We may agree that if statesmen (and leaders) cared for such studies as these, reasonable views might prevail outside the domain of logic ; the hypothesis is, however, equivalent to saying, " if leaders (and statesmen) were other than they are ", from which any conclusion might follow. Certainly the present reviewer much regrets that in Great Britain so few wish to study these exact and formal disciplines and still fewer are competent to teach them.

L. SUSAN STEBBING.

Non-euclidean Geometry. By H. S. M. COXETER. Pp. xv, 281. $3.25. 1942. Mathematical Expositions, 2. (University of Toronto Press : Humphrey Milford)

This second volume of the series " Mathematical Expositions " is learned, readable and attractive. All who are interested in geometry should browse over it. It is one of the mathematical books I should like to have in my pocket if Fate ever consigns me to a prison-camp (the present-day equivalent of a desert island). The plan of the book is as follows :

I. The historical development of non-euclidean geometry. II. Real projective geometry : foundations. III. Polarities, conics and quadrics. IV. Homogeneous coordinates. V, VI, VII. Elliptic geometry in one, two and three dimensions. VIII. Descriptive geometry. IX. Euclidean and hyperbolic geometry. X. Hyperbolic geometry in two dimensions. XI. Circles and triangles. XII. The use of a general triangle of reference. XIII. Area. XIV. Euclidean models.

The introductory chapters on real projective geometry are a model of clear, ordered and powerful reasoning which should make the connoisseur purr with pleasure. (Incidentally, a reform in the teaching of projective geometry in our schools and colleges is long overdue. The way is very clearly indicated in Coxeter's opening chapters. A textbook which has profoundly influenced geometrical teaching in the U.S.A., Veblen and Young's *Projective geometry*, has never " arrived " over here. Perhaps a book written in Canada by an English admirer of Veblen may meet with a happier fate.)

The various non-euclidean geometries are introduced synthetically and sympathetically and the reader soon finds himself led by his expert guide into brave new worlds constructed, as the references well show, by a truly inter-

national corps of mathematicians. On reaching the end of the book one feels that with its ample text, numerous references and excellent bibliography, it should be the standard textbook on non-euclidean geometry for a long time to come.

It is to be hoped that copies of this and other volumes in the series of "Mathematical Expositions" will soon be available in our bookshops. At the moment textbooks from across the Atlantic are evidently not given a high priority as cargo. There should be a little room between the cheeses for some mental food as well. If not, one would willingly sacrifice some of the cheese ration for the privilege of being able to read, on the rare occasions when there is a little energy left over from the common task, a book as stimulating and as satisfying as the work under review. D. PEDOE.

What is Mathematics? By R. COURANT and H. ROBBINS. Pp. xix, 521. 25s. 1942. (Oxford University Press)

In recent years there have been several attempts to answer the question adopted as a title by this book. Most of these attempts have contained good material. But some have endeavoured to gratify those half-jesting Pilates who want an answer in not more than an hour's light reading, while others have dilated on social, political and economic irrelevancies which, in this connection, are apt to generate more heat than light. Dr. Courant, in his preface, puts the matter squarely. He and his collaborator have striven to answer the question fairly and lucidly, but the reader must do his share in bringing hard work and serious, unremitting attention to his reading. The authors have found us a description ; they do not propose to relieve us of our duty of industrious thinking. That granted, they take great pains to make difficult points as simple as is consistent with the logical demands of the subject. "Technicalities and detours should be avoided and the presentation should be just as free from emphasis on routine as from forbidding dogmatism which refuses to disclose motive or goal. . . . It is possible to proceed on a straight road from the very elements to vantage points from which the substance and driving forces of modern mathematics can be surveyed." Nevertheless, "understanding of mathematics cannot be transmitted by painless entertainment."

Roughly, the book deals with the number systems of mathematics, geometrical construction, projective geometry and the beginnings of topology, functions and limits, maxima and minima (before the calculus, and so chiefly geometrical in style, but one of the best chapters in the book) and the calculus.

On the whole, the geometrical sections seem the best, if we include the chapter on maxima and minima already mentioned. Perhaps the reasons for this are, first that the technique is more easily expounded and more intuitively accepted than the corresponding analytical technique, and secondly that the results and theorems are often capable of simple expression in everyday language. A child of nine will grasp the point of the four-colour problem, and will draw endless Pascal diagrams for a circle or for two straight lines and gloat over the collinearities, but the prime-number theorem is far beyond him.

The chapter on geometrical constructions links this subject with number fields and so deals with possible and impossible constructions. Then turning to methods of construction, it deals with transformations, particularly inversion and the associated linkages of Peaucellier and Hart, and also with special or restricted methods of construction, such as the Mascheroni constructions using only a compass. In the next chapter, projective transformations lead to axiomatic structure of the subject, and also to ideal elements. There is a short but excellent section on non-euclidean geometry. Chapter V

is on topology, the four-colour problem, the Jordan curve, knots, Euler's formula for polyhedra and thence to the ideas of genus, connectivity and one-sided surfaces. Chapter VII again is extremely good; we start with quite simple geometrical problems in maxima and minima, for instance the reflected light ray problem. A general principle is then elucidated: if at a point R on a curve C a function $f(x, y)$ has an extreme value a, the curve $f(x, y)=a$ is tangent to C at R. This leads to the connection with the derivative, but throughout the chapter geometrical ideas are given prominence, and we end up, as we might expect from some of Courant's recent work, with Plateau's problem of finding the surface of smallest area bounded by a given curve in space, and the use of soap-film experiments in this connection.

The analytical chapters, four in all, are good, but on the whole more stereotyped and formal than the rest. A pretty paradox on p. 19 is worth noting, where by a fallacious induction argument it is proved that any two positive integers are equal. An interesting heuristic argument leading to the prime-number theorem, by means of processes similar to those employed in statistical theory, is given at the end of the calculus chapter.

In the preface Dr. Courant remarks that " under the pressure of other work, some compromise had to be made in publishing the book after many years of preparation, yet before it was really finished ". This explains a lack of polish in some places. Though the volume is beautifully printed, there are many misprints, not all of which are noted on the loose leaf correcting some 80 of them. But there is also an occasional rough edge in the argument. For example, careful attention is given to the method of mathematical induction, that *if* (i) the truth of the assertion A_r implies the truth of A_{r+1} and (ii) A_1 is true, *then* A_r is true for all r; and it is contrasted with empirical induction, proceeding from a particular series of observations of some phenomenon. As an application, it is proved that " by drawing n lines ", presumably straight, " we cannot divide the plane into more than 2^n parts ". But in the exposition, the reader might easily be led to think that the bound 2^n can be attained, and, what is more important, he will find the proof on the lines of " $n=1$, not more than 2 parts; $n=2$, not more than 4 parts, and so on ". Of course the proof is absolutely sound, but unfortunately it is so framed as to appear to be an instance of empirical rather than of mathematical induction. This is one of a number of instances in which the final haste of publication has allowed a slight blemish to remain.

Without any doubt, a book for keen pupils, thoughtful teachers, and for all school libraries. T. A. A. B.

The Laplace Transform. By D. V. WIDDER. Pp. x, 406. 36s. 1941. Princeton Mathematical series, 6. (Princeton University Press; Humphrey Milford)

The Laplace integral $$\phi(s)=\int_0^\infty e^{-st}f(t)\,dt, \qquad \text{(i)}$$

where s is a complex variable and $f(t)$ a real or complex function of the real variable t, sets up a transformation between functions $\phi(s)$, $f(t)$. This transformation has many uses in applied mathematics; in pure mathematics it can be used with effect to prove and to discover properties, mainly of formal type, of many special functions $f(t)$, such as the Bessel functions, the Laguerre and Hermite polynomials. But it can also be regarded as a well-defined domain of general function theory, in which general properties of the correlation are studied. The present volume is devoted to this last point of view, and the author completes and very considerably extends the treatment given in an earlier treatise by Doetsch.

One of the earliest questions to arise is that of convergence: for what values of s will the limit

$$\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,dt$$

exist? For suitably restricted $f(t)$ it can be shown fairly simply that the region of convergence is a half-plane, since if the integral converges for $s=\sigma_0+i\tau_0$ it will converge for all $s=\sigma+i\tau$ for which $\sigma>\sigma_0$. Thus the integral may converge for all s, for no s, or it may converge for s such that $\sigma>\sigma_c$ and diverge for s such that $\sigma<\sigma_c$. In the same way, it can be shown that there is a half-plane of absolute convergence, in general. Now this behaviour is precisely that of the Dirichlet series,

$$\phi(s)=\sum_1^\infty a_n \exp(-\lambda_n s), \qquad \text{(ii)}$$

and if we take as our definition of the transformation the Stieltjes integral

$$\phi(s)=\int_0^\infty e^{-st}\,d\alpha(t), \qquad \text{(iii)}$$

the original definition is recovered for an absolutely continuous function $\alpha(t)$, while if $\alpha(t)$ is a stretch-wise constant function the right-hand side becomes a Dirichlet series.

For this reason, Professor Widder takes (iii) as his fundamental transformation, where $\alpha(t)$ is a real or complex function of the real variable t, of bounded variation in any closed interval $0\leqslant t\leqslant T$. And again for this reason, the first chapter is an account of the Stieltjes integral. In the second chapter, we get the elementary theory of the Laplace integral, convergence, absolute convergence, uniform convergence, and the analytic character of the transform in its region of convergence. Chapter III deals with the moment problems of Stieltjes, Hausdorff and Hamburger; the latter problem includes the two former, since its object is to determine for what given sequences $\{\mu_n\}$ we can find a non-decreasing $\alpha(t)$ such that

$$\mu_n=\int_{-\infty}^{\infty} t^n\,d\alpha(t)\ldots(n=0,\,1,\,2,\,\ldots).$$

All this work links up in a very interesting way with summability questions.

After a not quite so interesting chapter on absolutely monotonic functions, Chapter V discusses Tauberian theorems, and the influence of Hardy and Littlewood, and Wiener, is very noticeable. Indeed, Professor Widder remarks in his preface that it was Hardy's Princeton lectures of 1928 which first interested him in the type of analysis contained in this book. The chapter ends very naturally with accounts of Wiener's and Ikehara's proofs of the prime number theorem by means of Tauberian theorems.

Chapter VI contains an account of the bilateral Laplace transform, in which the integral is taken from $-\infty$ to ∞; this is roughly equivalent to the well-known Mellin integral. Then in the next chapter we have some extremely interesting and very recent developments on the inversion of the Laplace integral, arising from Post's result that an inversion formula for equation (i) above is given by

$$f(t)=\lim_{k\to\infty}\frac{(-)^k}{k!}\left(\frac{k}{t}\right)^{k+1}\phi^{(k)}\left(\frac{k}{t}\right).$$

In the last chapter the Stieltjes transform is considered; this can be obtained formally as an iterative Laplace transform, leading to the definition

$$\phi(s)=\int_0^\infty \frac{d\alpha(t)}{s+t}.$$

Throughout the book, the analysis is step by step clear and precise. But in a treatise which opens up so many new fields for development perhaps more help could have been given to those students who will want to use the book to prepare themselves to carry its ideas further ; this help could have been given by more generous indications of the general lines of attack, particularly where the more difficult theorems are concerned, and also by a more frequent use of special functions to illustrate concretely the bearings of the general results. Doetsch's book shows us how effective at times such special illustrations can be. But it is no doubt ungracious to complain that Professor Widder has not given us more, when in 400 well-printed pages he has been able to present us with such a clear and up-to-date account of recent advances in a fascinating field of analysis. T. A. A. B.

Fourier Series and Orthogonal Polynomials. By DUNHAM JACKSON. Pp. xii, 234. \$2. 1941. (Mathematical Association of America)

This book, which is the sixth of the series of Carus Mathematical Monographs, deals with the theory of the expansion of an arbitrary function as a series of orthogonal functions, the Fourier Series being the simplest of this kind. Its scope is best indicated by a brief summary of its contents.

I. *Fourier Series* (pp. 1-44). The definition of a Fourier Series. Riemann's theorem on the limit of the Fourier constants ; Parseval's theorem. Sufficient conditions for convergence and uniform convergence, including Lebesgue's method of proving convergence for a function satisfying a Lipschitz condition. Fejér's theorem on summability $(C, 1)$. Weierstrass's approximation theorem.

II. *Legendre Polynomials* (pp. 45-65). Their derivation from a generating function ; elementary properties. Formal expansion of a function as a series of Legendre polynomials, and simple sufficient conditions for the convergence of such an expansion.

III. *Bessel Functions* (pp. 66-90). The definition of $J_0(x)$ from the differential equation. The zeros of $J_0(x)$ and the associated Fourier-Bessel expansion without discussion of convergence. Corresponding results for $J_n(x)$.

IV. *Boundary Value Problems* (pp. 91-114). The results of the preceding chapters are applied to the solution of boundary value problems for Laplace's equation in two dimensions and of initial value problems for the equation of wave motions in one and two dimensions.

V. *Double Series* : *Laplace Series* (pp. 115-141). This deals with the application of multiple Fourier series and series of spherical harmonics to boundary value problems in space.

VI. *The Pearson Frequency Functions* (pp. 142-148). The functions are discussed briefly because of the interest as weight functions for orthogonal polynomials.

VII. *Orthogonal Polynomials* (pp. 149-165). Schmidt's process for constructing a sequence of orthogonal polynomials corresponding to an arbitrary weight function. The Christoffel-Darboux identity. The case when the weight function is a Pearson Frequency Function.

VIII. *Jacobi Polynomials* (pp. 166-175). IX. *Hermite Polynomials* (pp. 176-183). X. *Laguerre Polynomials* (pp. 184-190).

These three chapters give a clear account of the elementary properties of these special types of orthogonal polynomial.

XI. *Convergence* (pp. 191-208). A discussion of simple sufficient conditions for the convergence of the expansion of an arbitrary function as a series of orthogonal polynomials in the case when the corresponding normalised orthogonal polynomials form a uniformly bounded set.

The book concludes with a set of exercises intended to illustrate and extend the text.

It is assumed that the reader knows a reasonable amount of calculus and a little about the Gamma function and easy differential equations, and also that he possesses or acquires in the course of reading the monograph a certain amount of " mathematical maturity ". The author remarks that " Under the circumstances, ' rigor ' in the sense of literal completeness of statement has been out of the question. It is hoped, however, that the reader who is familiar with the methods of rigorous analysis will be able without any difficulty to read between the lines the requisite supplementary specifications, and will find that what has actually been said is entirely accurate in the light of such interpretation ".

The task which the author has set himself is no easy one, but he has succeeded in producing a sound and very readable book well suited to the purpose he had in view. E. T. COPSON.

Elementary Mathematics. By HYMAN LEVY. Pp. 216. 5s. 1942. (Nelson)

Professor Levy has written in most attractive style a logical, precise and clear account of simple Arithmetic, Algebra, and some allied portions of elementary Mathematics. The question must, however, be asked : for whom is the book intended? The volume is one of a series of " Aeroscience Manuals ", presumably for students who are at the beginning of their careers. Experience with A.T.C. cadets leads to the feeling that those who need to study in detail and at slow pace the careful work on, for example, fractions, decimals or negative numbers (the contents of Chapters 2, 3 and 5 respectively) will be a very, very long time before being able to tackle such sections as Chapters 16 and 17, which deal with Rates of Change and Summation of Areas. Correspondingly those who will be able to appreciate the latter portions will probably be unwilling to spend time on the earlier, however beneficial such a course might be.

Topics intervening between those already mentioned include Statistical Examination of Data, Some Simple Facts of Geometry, First Steps in Algebra, Equations, " What is a Function? ", Some Special Functions (such as $\sum_1^n n^m$, and 10^x) and Problems of Calculation (which introduces discussions of $(1+1/x)^x$ and of " Infinity ").

The book can be recommended to any earnest student, but with the recommendation there could well be given a suggestion of the sections to be studied, selected according to the ability of the individual concerned. Every teacher should see it to realise how valuable can be a fresh presentation of familiar work. F. W. K.

Survey of the theory of integration. By J. DOUGLAS. Pp. 47. 50 cents. 1941. (*Scripta Mathematica*, Yeshiva College, New York)

This pamphlet is in the main historical and gives in outline various integrals, Riemann, Stieltjes and Lebesgue. Denjoy totalisation is barely mentioned. Six pages are devoted to the Stieltjes integral. P. J. D.

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